Classifications of Irregular Connections of One Variable

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1. Introduction

Let $\mathcal{O} = \mathbb{C}\{x\}$ be the space of germs of holomorphic functions at the origin, and $K = \mathcal{O}[x^{-1}]$ its field of fractions. We will denote by $\hat{\mathcal{O}}$ and $\hat{\mathcal{K}}$ their respective completions. Let E be a finite dimensional vector space over K and ∂ a connection on E, i.e. a $\mathbb{C}$-linear map $\partial\colon E \to E$ satisfying $\partial(\varphi e) = \dfrac{d\varphi}{dx}e + \varphi\partial e$ for $\varphi \in K$, $e \in E$. Recall that ∂ is said to be "regular" if there exists a basis $(e_1, \cdots, e_n)$ of E over K in which the matrix M of the connection (defined by $\partial e_i = \sum m_{ij}e_i$) has a simple pole. Otherwise, ∂ is said to be "irregular".

The classification in the regular case is assumed to be known. I propose here to explain the irregular case. The principle of this classification goes back to the fundamental paper of Birkhoff [Bir13], too long ignored except by a small number of specialists. In fact, Birkhoff treats the case where, in a suitable basis, the most polar part of M has distinct eigenvalues; on this case, see also [BJL79]. In the general case, a detailed study can be found in Jurkat [Jur78]. I will give here a version of the classification due to Deligne [Del], which relies on previous remarks from [Sib77] and [Mal79]. In all these methods, an essential ingredient is a theorem on holomorphic invertible matrix functions by Sibuya [Sib90], a variant of which is already essentially found in [Bir13].

> I have to apologize for having delayed so long in writing an exposition on these questions, and also for the impossibility in which I find myself of giving the totality of the bibliographical references on this subject, references which should begin at least at Poincaré, or even Laplace.

2. Formal Classification

If L is a finite extension of K, there exists $t \in L$ and p a positive integer such that $t^p = x$, and $L = \mathbb{C}\{t\}[t^{-1}] = K[t]$. The given connection ∂ on E extends in a unique way to $E \otimes L = F$ by $(t\partial_t)(t^k \otimes e) = k(t^k \otimes e) + \dfrac{1}{p}(t^k \otimes (x\partial_x)e)$. If $\alpha \in L \otimes dt$, we will denote by F^α the L-vector space of rank 1 endowed with the connection defined by $\partial_t f = \dfrac{\alpha}{dt}f$. It is classical that F^α is isomorphic to F^β if and only if $\alpha - \beta$ has a simple pole, with the coefficient of t^{-1} being an integer.

The following theorem is classical (Fabry, Hukuhara, Turrittin; I don't know where the first complete proof can be found; one can find it in [Was87] and more recent ones in [Lev75], [Mal72], and [Rob80]).

2.1 Theorem. Let (E, ∂) be a vector bundle with connection over K. After possibly a ramification $t^p = x$, one can find a formal isomorphism

$$E \otimes_{\hat{K}} \hat{L} \cong \bigoplus_\alpha (F^\alpha \otimes_{\hat{L}} G^\alpha),$$

where the F^α have the meaning given above (with L replaced by $\hat{L}$), and where the G^α are regular.

By decomposing the G^α according to their indecomposable factors, one then obtains the indecomposable factors of $E \otimes L$. This decomposition is unique in the sense of the Krull-Schmidt theorem.

3. Asymptotic Expansions

Let (E, ∂) and (E', ∂) be two vector bundles with connection over K, and let $\hat{\alpha}\colon (\hat{E}', \partial) \xrightarrow{\sim} (\hat{E}', \partial)$ be an isomorphism of their completions. If E, and therefore E', is regular, we know that $\hat{\alpha}$ comes from an isomorphism $\alpha\colon (E', \partial) \to (E, \partial)$. This is no longer true in general if E is not regular; more precisely, one can see that there exist $\hat{\alpha}$ that do not descend if and only if $(\mathrm{End}_K(E), \partial)$ is irregular.

To obtain an analytic classification, one must therefore introduce other invariants, called "analytic invariants"; a first version of these invariants ([S], [Mal79]) involves sectorial asymptotic expansions, which are defined as follows:

We work in a neighborhood of $0 \in \mathbb{C}$; we perform a real blowup of 0, i.e. we pass to polar coordinates $(\rho, \theta) \in \mathbb{R}_+ \times T$; we denote by S the inverse image $\{0\} \times T$ of 0, and we construct a sheaf $\mathcal{A}$ on S as follows:

Let U be an open subset of S, and $\tilde{U}$ the associated angular sector of $\mathbb{C}$, i.e. $\{(\rho, \theta) | \rho > 0, \theta \in U\}$; let $\overline{\mathcal{A}}(U)$ be the set of germs at 0 of holomorphic functions f in $\tilde{U}$, admitting at 0 a Laurent asymptotic expansion (I am taking here a slightly different notation from that of [Mal79]); more precisely, we require that there exists a formal series $\sum_{n \leq n_0} a_n x^n \in \hat{K}$ such that, for all $p \in \mathbb{Z}$, and x close to 0,

$$\left| f(x) - \sum_{n \leq p} a_n x^n \right| \leq C_p \left| x^{p+1} \right|, \; C_p > 0.$$

A classical theorem of Ritt ensures that, if $U \neq S$, the "Laurent series" map $\overline{\mathcal{A}}(U) \to L$ is surjective (see e.g. [Was87]); in what follows, we will denote this map by $T\colon f \mapsto \tilde{f}$.

3.1 Definition. We denote by $\mathcal{A}$ the sheaf associated to the presheaf $U \mapsto \overline{\mathcal{A}}(U)$.

With this in place, the first result on which we will rely is the following fundamental theorem.

3.2 Theorem (Hukuhara-Turrittin). Let (E,∂) be a vector bundle with connection over K. Then, for all $\theta \in S$, the map $T\colon \ker(\partial, \mathcal{A}_\theta \otimes_K E) \to \ker(\partial, \hat{E})$ is surjective.

A proof of this theorem can be found in [Was87][1].

Note that the usual statements are apparently stronger, since one proves the previous result in any sector of opening $< \pi$, π being the Katz invariant of (E,∂). In fact, it is known that cohomological arguments combined with the formal theory explained in §2 suffice to recover this result from (3.2). We will see arguments of this type in §5.

Let us now fix an (E,∂). We will use the previous result to study the vector bundles with connection (E',∂) endowed with an isomorphism of the completions $\hat{\alpha}\colon (\hat{E}',\partial) \xrightarrow{\sim} (\hat{E},\partial)$ (we follow here the reasoning of [Mal79]); for this, we apply the previous theorem to $\hat{\alpha}$ considered as a horizontal section of $\hom_{\hat{K}}(\hat{E}',\hat{E}) = \hom_K(E',E)^\wedge$; there thus exists a covering $\{U_i\}$ of S such that, in U_i, $\hat{\alpha}$ is represented by α_i, a horizontal section over U of $\hom_{\mathcal{A}}(\mathcal{A}\otimes_K E', \mathcal{A}\otimes_K E)$; since $\hat{\alpha}_i = \hat{\alpha}$ is invertible, we easily deduce that α_i exists. Then, for all (i,j), $\alpha_i\alpha_j^{-1} = \beta_{ij}$ is an invertible horizontal section over $U_i \cap U_j$ of $\mathrm{End}_{\mathcal{A}}(\mathcal{A}\otimes_K E) = \mathcal{A}\otimes_K \mathrm{End}_K(E)$; moreover, as $\hat{\alpha}_i = \hat{\alpha}_j$, we have $\hat{\beta}_{ij} = \mathrm{Id}$.

We then denote by $\Lambda(E)$ the sheaf of invertible sections of $\mathcal{A}\otimes_K \mathrm{End}_K(E)$; by taking a basis $e_1,\cdots,e_n$ of E, $\Lambda(E)$ identifies with the sheaf $G\ell(n,\mathcal{A})$ of invertible matrices with coefficients in $\mathcal{A}$. Let $\Lambda_0(E)$ be the subsheaf of elements of $\Lambda(E)$ asymptotic to the identity, and $\Lambda_0(E,\partial)$ the subsheaf of horizontal sections for ∂ of $\Lambda_0(E)$. What precedes gives a cocycle $\{\beta_{ij}\}$ of $\Lambda_0(E,\partial)$ for the covering $\{U_i\}$, from which we obtain by passing to the quotient a cohomology class $\gamma(\hat{\alpha}) \in H^1(S,\Lambda_0(E,\partial))$; one easily verifies that $\gamma(\hat{\alpha})$ depends only on $\hat{\alpha}$, and not on the chosen covering and liftings α_i.

Let us say on the other hand that $(E',\partial,\hat{\alpha})$ and $(E'',\partial,\hat{\alpha}')$ are equivalent if the isomorphism $\hat{\alpha}'^{-1}\hat{\alpha}\colon (\hat{E}',\partial) \to (\hat{E}'',\partial)$ comes from an isomorphism (necessarily unique) $(E',\partial) \to (E'',\partial)$. We then have the following result:

3.3 Lemma. $(E',\partial,\hat{\alpha})$ and $(E'',\partial,\hat{\alpha}')$ are equivalent if and only if $\gamma(\hat{\alpha}) = \gamma(\hat{\alpha}')$.

Proof. Suppose we have $\gamma(\hat{\alpha}) = \gamma(\hat{\alpha}')$. By refining the coverings if necessary, we may assume that the $\hat{\alpha}$ and $\hat{\alpha}'$ are defined on the same covering $\{U_i\}$ and that there exist $\beta_i \in \Gamma(U_i,\Lambda_0(E,\partial))$ such that on $U_i\cap U_j$, $\hat{\alpha}'_i\hat{\alpha}'^{-1}_j = \beta_i\alpha_i\alpha_j^{-1}\beta_j^{-1}$; we then have $\alpha'^{-1}_j\beta_j\alpha_j = \alpha'^{-1}_i\beta_i\alpha_i$; these functions glue together into a global section on S of $\mathcal{A}\otimes\hom_K(E',E'')$, a section which will necessarily be meromorphic, so will belong to $\hom_K(E',E'')$; moreover, δ will obviously be invertible, and will satisfy $\delta = \hat{\alpha}'^{-1}\hat{\alpha}$ on passing to asymptotic expansions; hence $(E',\partial,\hat{\alpha})$ and $(E'',\partial,\hat{\alpha}')$ are equivalent. The converse is proved similarly. □

Finally, let $C\ell(E,\partial)$ denote the set of $(E',\partial',\hat{\alpha})$ up to equivalence; what precedes gives an injective map $\gamma\colon C\ell(E,\partial) \to H^1(S,\Lambda_0(E,\partial))$

3.4 Theorem. The map $\gamma\colon C\ell(E,\partial) \to H^1(S,\Lambda_0(E,\partial))$ is bijective.

It remains to prove surjectivity. It follows from the following theorem.

3.5 Theorem. The map $H^1(S,\Lambda_0(E)) \to H^1(S,\Lambda(E))$ has image zero.

This result is due to Sibuya [Sib90]; a proof will be given in the appendix.

Let us show how this result implies (3.4). Let $\beta \in H^1(S,\Lambda_0(E,\partial))$; for a suitable covering $\{U_i\}$ of S, β is represented by $\beta_{ij} \in \Gamma(U_i \cap U_j, \Lambda_0(E,\partial))$; according to (3.5) we

[1] I should note on this subject that I do not know if the results that I imprudently announced without proof at the end of [Mal72] and [Mal79] are true in full generality.

can write $\beta_{ij} = \alpha_i\alpha_j^{-1}$, with $\alpha_i \in \Gamma(U_i, \Lambda(E))$. Then endow $\mathcal{A}\otimes_K E|U_i$ with the connection $\alpha_i^{-1}\partial\alpha_i \underset{\text{def}}{=} \partial'$; on $U_i \cap U_j$, we have $\partial' = \partial$ since $\beta_{ij} = \alpha_i^{-1}\alpha_j$ is horizontal; hence these connections glue together to give a connection ∂' on E'. Moreover, we have $\hat{\alpha}_i = \hat{\alpha}_j$, so the $\hat{\alpha}_i$ define an isomorphism $\hat{\alpha}\colon \hat{E} \to \hat{E}$; it is then clear by construction that $\hat{\alpha}$ is an isomorphism $(\hat{E}, \partial') \to (\hat{E}, \partial)$, and that we have $\gamma(\hat{\alpha}) = \beta$; whence the theorem.

4. Stokes Structures

We will now translate the results of the previous paragraph in terms of asymptotic expansions of the sectorial solutions of the equations considered; I follow here Deligne [Del].

Let (E, ∂) be a vector bundle with connection over K. Let V be the locally constant sheaf on S of sectorial horizontal sections of E, defined as follows: for $\theta \in S$, V_θ is the space of horizontal sections of (E, ∂) over a small sector $\{0 < |x| < \varepsilon\} \cap \{|\arg x - \theta| < \varepsilon\}$.

Apply Theorem (2.1): after possibly a ramification $t^p = x$, we can find a formal isomorphism $\hat{\lambda}\colon \hat{E}\otimes_K L \to \hat{E}_1$ where $\hat{E}_1$ is of the form $\bigoplus_{\alpha\in A}(F^\alpha\otimes_L G^\alpha)$; by applying Theorem (3.2) to $\hom_L(E \otimes_K L, E_1)$, we obtain a sectorial isomorphism u_θ in a neighborhood of θ: $E \otimes_K L \to E_1$, given by an invertible element of $\mathcal{A}_\theta \otimes_L \hom(E \otimes_K L, E_1)$, which will therefore transform V_θ into $V_{1,\theta}$ ($V_{1,\theta}$ being the local system of horizontal sections of E_1). Moreover, $V_{1,\theta}$ is immediately explicit: the sections of E_1 are of the form $\sum_{\alpha\in A} e^{-\int\alpha} f_\alpha$, where f_α is a solution of an equation with regular singularities; by u_θ, we deduce the asymptotic behavior of the horizontal sections of (E, ∂) in a sector near θ; in particular, we can put a partial order on V_θ according to which exponentials intervene in the said asymptotic behavior. This leads to the following construction.

Let I be the following local system on S: over a sector we take the forms $\sum_{-n}^{+\infty} a_k x^{k/p} dx$ (p any positive integer), modulo poles of order ≤ 1.

On I, we define the following partial order: for $\theta \in S$, we have $\alpha <_\theta \beta$ if $e^{-\int(\alpha-\beta)}$ is slowly growing (i.e. $O(|x|^{-N})$ for some $N > 0$) in a small sector around θ. Note that, for given α and β, $\alpha \neq \beta$, there exists a finite number of points θ of S (or more exactly, of a finite covering of S) such that α and β are incomparable in the neighborhood of θ; in this case, for θ' near θ on one side, we will have $\alpha <_{\theta'} \beta$; on the other side, we will have $\beta <_{\theta'} \alpha$ (we write $<$ for $\leq$ and $\neq$). The corresponding half-lines are traditionally called the "Stokes lines" relative to (α, β).

4.1 Definition. Let V be a local system (= a sheaf locally isomorphic to $\mathbb{C}^n$) on S. A Stokes structure, or I-filtration of V is a family of subsheaves V^α, indexed by I, satisfying the following property:

For all $\theta \in S$, there exists a decomposition $V_\theta = \bigoplus V_{\alpha,\theta}$ such that for all θ' near θ

$$V_{\theta'}^\alpha = \bigoplus_{\beta\leq_{\theta'}\alpha} V_{\beta,\theta}.$$

(Beware that the V^α are not subsheaves in the usual sense, since they are indexed by a local system and not a set).

We define $\operatorname{Gr} V$ by $(\operatorname{Gr} V)^\alpha_\theta = \bigoplus V^\alpha_\theta \Big/ \sum_{\beta <_\theta \alpha} V^\beta_\theta$; the property (4.1) ensures that the $(\operatorname{Gr} V)^\alpha$ form a family of local systems indexed by I(same warning as above).

With this in place, let (E, ∂) be a vector bundle with connection over K, and V the local system of its solutions; the construction at the beginning of this paragraph provides a Stokes structure on V, which we can further restrict to indexing by the α that intervene in the decomposition of E_1, the others playing no role.

What precedes gives a functor

$$\Phi\colon (\text{vector bundles with connection over } K) \to (I\text{-filtered local systems}),$$

the map on "Hom" being evident. The result is then the following.

4.2 Theorem. Φ is an equivalence of categories.

Proof. A) Let us first show that Φ is fully faithful. For this, consider two vector bundles with connection (E, ∂) and (E_1, ∂), and let $F = \hom_K(E, E_1)$, endowed with ∂; set $V = \Phi(E, \partial)$, $V_1 = \Phi(E_1, \partial)$, $W = \Phi(F, \partial)$; one immediately verifies that, if we denote by $\bar{V}$ the local system V where we have forgotten the filtration, we have $\bar{W} = \underline{\hom}(\bar{V}, \bar{V}_1)$, and that moreover W is endowed with the filtration defined by the fact that W^α maps V^β into $V_1^{\alpha+\beta}$ for all β. In particular, $\hom(V, V_1)$ identifies with the sections of W^0, i.e. the meromorphic horizontal sections of $\hom_K(E, E_1)$, which gives the desired result.

B) To prove that Φ is essentially surjective, we need to introduce another functor $\hat{\Phi}$ which we will now define.

4.3 Lemma. Let $(\hat{E}, \partial)$ be a vector bundle with connection over K; there exists (E_1, ∂) over K whose completion is isomorphic to $(\hat{E}, \partial)$.

Take a basis of $\hat{E}$, say $(e_1, \cdots, e_n)$ and let M be the matrix of ∂ in this basis; the change of basis $(e_1, \cdots, e_n) = (f_1, \cdots, f_n)S$ transforms M into N, satisfying

$$N = SMS^{-1} - \frac{dS}{dx}S^{-1}, \text{ or equivalently } \frac{dS}{dx} = SM - MS;$$

in this situation, we will say that N is equivalent to M; if moreover, S is of the form $\operatorname{Id} +$(terms of order > 0), we will say that N is strictly equivalent to M.

The lemma is a consequence of the following result: any N sufficiently close to M, i.e. such that $N - M$ is of order $\gg 0$, is strictly equivalent to M.

It suffices to establish this result after a suitable ramification $t^p = x$; indeed, to go back to the initial situation, it will suffice to keep in the matrix S obtained the integral powers of x. The result is then proved at the same time as the formal reduction (2.1); see on this subject the calculations of [Rob80].

Let then $(\hat{E}, \partial)$ be a $\hat{K}$-vector bundle with connection, and let (E_1, ∂) over K, endowed with an isomorphism $\lambda_1\colon (\hat{E}, \partial) \to (\hat{E}_1, \partial)$. We set $\hat{\Phi}(\hat{E}, \partial) = \operatorname{gr}\Phi(E_1, \partial)$; if we have another system $(E_2, \partial, \lambda_2)$, with $\lambda_2\colon (\hat{E}, \partial) \xrightarrow{\sim} (\hat{E}_2, \partial)$, we have a well-defined isomorphism $\operatorname{gr}\Phi(E_1, \partial) \xrightarrow{\sim} \operatorname{gr}\Phi(E_2, \partial)$ defined as follows: in a sufficiently small sector U, $\lambda_2\lambda_1^{-1}$ is represented by a horizontal section μ of

$\mathcal{A}(U) \otimes_K \hom_K(E_1, E_2)$, whence a map $V_1 \to V_2$ over U ($V_i = \Phi(E_i, \partial)$); if we change μ to μ', then $\mu' - \mu$ is asymptotic to 0, i.e. belongs to $\underline{\hom}(V_1, V_2)^{<0}$, so induces 0 on the associated graded objects. Hence $\hat{\Phi}(\hat{E}, \partial)$ does not depend on (E_1, ∂). We define the map on "Hom" by the same process. Finally, we obtain a commutative diagram of functors:

$$\begin{array}{ccc} (\text{vector bundles with connection over } K) & \xrightarrow{\Phi} & (I\text{-filtered local systems}) \\ \big\downarrow{\scriptstyle \text{formalize}} & & \big\downarrow{\scriptstyle \text{gr}} \\ (\text{vector bundles with connection over } \hat{K}) & \xrightarrow{\hat{\Phi}} & (I\text{-graded local systems}) \end{array}$$

C) We will first prove the following theorem

4.4 Theorem. $\hat{\Phi}$ induces an equivalence of categories.

The fact that $\hat{\Phi}$ is fully faithful is seen easily, by the same type of arguments as for Φ. It remains to prove that $\hat{\Phi}$ is essentially surjective.

Let V be an I-graded local system; if the $\alpha \in I$ for which $V_\alpha \neq 0$ are unramified, the result is immediate; it suffices to take $E = \bigoplus (F^\alpha \otimes_K G^\alpha)$, the F^α having the same meaning as in Theorem (2.1), and the G^α being regular singular with monodromy equal to that of V^α.

In the general case, let p be such that, after the change of variable $t^p = x$, the α for which $V^\alpha \neq 0$ are unramified; let T be the covering of degree p of S and $\pi\colon T \to S$ the projection. The resulting $\pi^*(V)$ is represented by a vector bundle with connection $(\hat{F}, \partial)$ over $\hat{K}[t] = \hat{L}$. Since $\hat{\Phi}$ is fully faithful, the action of the Galois group $\mathrm{Gal}(T/S) = \mathrm{Gal}(L/K)$ gives an action of $\mathrm{Gal}(L/K)$ on $(\hat{F}, \partial)$; one sees easily that it suffices to take the invariants to represent V. Whence Theorem (4.4).

D) Let us finally show that Φ is essentially surjective; for this, it suffices to remark the following: let V be an I-graded local system; by (4.3) and (4.4) we can already assume that there exists an (E_1, ∂) over K, with $V_1 = \Phi(E_1, \partial)$, such that $\operatorname{gr} V_1$ is isomorphic to $\operatorname{gr} V$. Hence, it suffices to see that Φ is a bijection between $C\ell(E_1, \partial)$ (notations of Theorem (3.4)) and the I-filtered local systems V' endowed with an isomorphism $\operatorname{gr} V' \xrightarrow{\sim} \operatorname{gr} V_1$. But, the said systems are classified by $H^1(S, \underline{\mathrm{Aut}}_0(V_1))$, denoting by $\underline{\mathrm{Aut}}_0(V_1)$ the sheaf of automorphisms of V_1 which induce the identity on the associated graded. Moreover, $\underline{\mathrm{Aut}}^0(V_1)$ is the sheaf of sections of $W = \Phi(\mathrm{End}_K(E_1), \partial)$ which are of the form $\mathrm{Id} + \lambda$, with $\lambda \in W^{<0}$; this sheaf is therefore equal to $\Lambda_0(E_1, \partial)$ and we conclude by Theorem (3.4). □

5. An Example

To make the previous constructions more concrete, and also to prepare a later exposition, we will look explicitly at the classification of vector bundles with connection over K which are formally isomorphic to $E = \bigoplus F^\alpha \otimes_K G^\alpha$, $\alpha \in A \subseteq I$, $\alpha = \sum_{-r}^{0} a_k(\alpha) x^{k-1} dx$ ($r \geq 1$ given), with G^α having regular singularities and $a_{-r}(\alpha)$s distinct for the various

α. We will follow here the method of [BJL79]; a different method can be found in Birkhoff [Bir13]; this last one was extended to the general case by Jurkat [Jur78].

Let $V = \Phi(E)$; we have here a decomposition $V = \bigoplus V_\alpha$, $V_\alpha = \Phi(F^\alpha \otimes G^\alpha)$, i.e. a canonical lifting $\operatorname{gr} V \to V$. Let W be an A-filtered local system, endowed with an isomorphism $\hat{\lambda}$: $\operatorname{gr} W \to \operatorname{gr} V$.

The Stokes lines are here the half-lines on which $\operatorname{Re}[(a_{-r}(\alpha) - a_{-r}(\beta))x^{-r}] = 0$; for each pair (α,β), $\alpha \neq \beta$, we thus have $2r$ such half-lines, each making with the preceding one an angle $\dfrac{\pi}{r}$; we will denote them by $D^k_{\alpha\beta}$, $k = 1, \cdots, 2r$. We do not exclude the case where two such lines, corresponding to distinct pairs, are confounded. We will call an open interval $U \subseteq S$ (or the corresponding sector) "good" if it has the following property: for any pair (α,β), U intersects one and only one of the half-lines $D^1_{\alpha\beta}, \cdots, D^{2r}_{\alpha\beta}$.

There obviously exist good intervals (take any interval of length $\dfrac{\pi}{r}$ whose endpoints do not belong to any Stokes line modulo 2π, and slightly enlarge the previous intervals).

5.1 Lemma. For each U, there exists a unique lifting $\lambda(U)$: $W|U \to V|U$ of $\hat{\lambda}$: $\operatorname{gr} W \xrightarrow{\sim} \operatorname{gr} V$.

Proof. The uniqueness of $\lambda(U)$ is obvious: indeed, since one of the lines $D^k_{\alpha\beta}$ meets U, whatever the pair (α,β), $\alpha \neq \beta$, α and β are globally incomparable on U; it follows that the only automorphism of $V|U$ that induces the identity on $\operatorname{gr} V$ is the identity.

To prove existence, take an open interval $U_1 \subseteq U$ and a lift $\lambda(U_1)$: $W|U \to V|U$ of $\hat{\lambda}$ (this exists by Theorem (3.2)), and let θ be an endpoint of U_1; if $\theta \notin U$, it is done; otherwise there are two cases to consider.

First case.

θ does not belong to a Stokes line. We will see that then λ extends beyond θ, which allows us by connectivity to reach the next Stokes line.

Indeed, let U_1 be a small interval around θ, not meeting any Stokes line, and take a lift $\lambda(U_2)$: $W|U_2 \xrightarrow{\sim} V|U_2$ of $\hat{\lambda}$. Number $\alpha_1 < \alpha_2 < \cdots < \alpha_p$ by the order of the αs in U_2, with $p = \operatorname{card} A$.

Let e_α be a basis of V_α over $U_1 \cup U_2$; set $f_\alpha = \lambda(U_1)^{-1} e_\alpha$, $g_\alpha = \lambda(U_2^{-1}) e_\alpha$; on $U_1 \cap U_2$ we have the relations $f_{\alpha_i} = g_{\alpha_i} + \sum_{j<i} g_{\alpha_i} m_{ji}$, m_{ij} constant matrices; it follows that on U_2, we still have $f_{\alpha_i} \in W^{\alpha_i}$, whence the desired result.

Second case.

θ belongs to a Stokes line; let U_2 be a small interval around θ, not meeting any other Stokes line; we will see that we can find another lift $\lambda'(U_1)$ of $\hat{\lambda}$ that extends to $U_1 \cup U_2$. Combining with the 1st case, we will ultimately obtain the result.

Note again $\alpha_1 < \cdots < \alpha_p$ the order of the α in a neighborhood U_1 of θ. At a point $\theta' \in S$, the order of the α is given by $\operatorname{Re}(a_{-r}(\alpha)x^{-r})$, $\arg x = \theta'$, the distinct αs for which this expression is equal being incomparable at θ'; it follows that there exist in $\{1, \cdots, p\}$ disjoint intervals $I_1, \cdots, I_s$ such that for θ' close to θ, $\theta' \notin U_1$, with the order of the α_i as follows:

i) in each interval I_j, the initial order (= in U_1, near θ), is reversed;

ii) all other order relations are preserved.

Choose then a lift $\lambda(U_2)\colon W|U_2 \xrightarrow{\sim} V|U_2$, and let f_α, g_α be defined as in the first case. On $U_1 \cap U_2$, we still have

$$f_{\alpha_i} = g_{\alpha_i} + \sum_{j<i} g_{\alpha_j} m_{ji}; \tag{5.2}$$

We modify the lift $\lambda(U_1)$ to $\lambda'(U_1)$ as follows:

- If $i \notin I_1 \cup \cdots \cup I_s$, we take $\lambda'^{-1}(U_1)e_{\alpha_i} \underset{\text{def}}{=} f'_{\alpha_i} = f_{\alpha_i}$.
- If i belongs to one of the I_k, we take:

$$f'_{\alpha_i} = f_{\alpha_i} + \sum_{j<i,\ j\in I_k} f_{\alpha_j} n_{ji}. \tag{5.3}$$

This indeed gives a lift of $\hat{\lambda}$ over U_1 whatever the n_{ij} are chosen, since U_1 does not meet by hypothesis any Stokes line relative to the pairs (i, j) belonging to the same interval I_k. Now, combining (5.2) and (5.3), we verify that there exists a unique choice of the n_{ij} for which we still have, on U_2: $f'_{\alpha_i} \in W^{\alpha_i}$, $i = 1, \cdots, p$. This proves the lemma. □

We will say that an open cover $\{U_1, \cdots, U_{2r}\}$ of S is "good" if it has the following properties:

i) all the U_is are good;

ii) U_i meets only U_{i-1} and U_{i+1} (we set $U_{2r+1} = U_1$);

iii) $U_i \cap U_{i+1}$ does not contain any Stokes line.

We can always find good covers (take the closed cover of S by the interval $[\theta_0 + k\pi/r, \theta_0 + (k+1)\pi/r]$, θ_0 being chosen distinct from the Stokes directions modulo π/r, and slightly enlarge the previous intervals). For each U_i, there exists a unique lift $\lambda(U_i)\colon W|U_i \to V|U_i$ of $\hat{\lambda}$. It is then clear that the Stokes structure is given by the choice of the $\lambda(U_i)\lambda^{-1}(U_{i+1})$; these are automorphisms of $V|U_i \cap U_j$ inducing the identity on the associated graded; under this sole restriction, their choice is arbitrary. For $r \geq 2$, $U_i \cap U_j$ is a sector; with respect to the decomposition $V = \oplus V_\alpha$, $\lambda(U_i)\lambda^{-1}(U_{i+1})$ is expressed by a strictly triangular matrix with respect to the order of the αs in $U_i \cap U_{i+1}$; if $r = 1$, I leave the reader to adapt. Finally, by taking bases of the V_α over $U_i \cap U_{i+1}$, we obtain an isomorphism $C\ell(E, \partial) \simeq \mathbb{C}^N$, with $N = r \sum_{\alpha\neq\beta} \dim V_\alpha \cdot \dim V_\beta$.

We immediately verify that N is the irregularity in the sense of [Mal72] of $(\mathrm{End}_K E, \partial)$; this property extends to the general case, treated in [Jur78].

5.4 Remark. If we change the cover (and the bases of the V_α), we obtain an automorphism of $\mathbb{C}^N$ which we can see is polynomial. So, in fact, $C\ell(E, \partial)$ is naturally endowed with an affine space structure of dimension N. As this will be useful in the promised later exposition, I will sketch the proof. It suffices to see this: let U' be a good open set and $\lambda(U')\colon W|U' \to V|U'$ the lift of $\hat{\lambda}$ given by (5.1). Then, for every i such that $U' \cap U_i \neq 0$, $\lambda(U')\lambda(U_i)^{-1}$ has, in a basis of V, polynomial coefficients with respect to those of $\lambda(U_i)\lambda^{-1}(U_{i+1})$. The only non-trivial case is when, for some i, we have $U' \subseteq U_i \cup U_{i+1}$, $U' \not\subset U_i$, $U' \not\subset U_{i+1}$ (otherwise, it is easy to see that, for some j: $U' \subset U_j$ or $U' \supset U_j$); we then write the analogous equations to (5.2) and (5.3) for

$\lambda(U')\lambda(U_i)^{-1}$ and $\lambda(U')\lambda(U_{i+1})^{-1}$ and the relations between them, given by the coefficients of $\lambda(U_i)\lambda^{-1}(U_{i+1})$; we easily conclude from the fact that the obtained equations have a unique solution for every choice of $\lambda(U_i)\lambda^{-1}(U_{i+1})$.

6. Remarks on a Moduli Problem

We will keep the example of §5 (what we will say would generalize by using [Jur78]). Let D be an open disk of $\mathbb{C}$ centered at 0, and T an analytic complex variety; denote by Z the zero section $T \times \{0\}$ of $T \times D$, and by K the sheaf of meromorphic functions on $T \times D$ with poles in Z. We will call a "family of vector bundles with connection on D, parametrized by T" a free K-module of finite type L endowed with a partial derivation $\frac{\partial}{\partial x}$ (x the variable of D). Working with the formal completion of K along Z, we similarly define "formal families of vector bundles with connection, parametrized by T". We call a "family of vector bundles with connection on D, endowed with a formal isomorphism with E" a family $\left(L, \frac{\partial}{\partial x}\right)$ whose formalization $\left(\hat{L}, \frac{\partial}{\partial x}\right)$ along Z is isomorphic to the constant family defined by $(\hat{E}, \partial)$. We may wonder if there exists a moduli space for these families, whose base would be $C\ell(E,\partial)_{an}$, i.e. $C\ell(E,\partial)$ endowed with the analytic affine structure that has just been defined.

We can see that the answer is positive; as the problem is of limited interest, I will only say a few words about it. First of all, to see that $C\ell(E,\partial)an$ is a coarse moduli space, it suffices to see that every family of this type gives rise canonically to an analytic map $T \to C\ell(E,\partial)_{an}$; this is seen by using a theorem of Sibuya [Sib68] which "puts parameters" into (3.2). To construct a universal family over $C\ell(E,\partial)_{an}$, we essentially need to put parameters into (3.5), which presents no difficulty, and to use Grauert's theorem which will tell us that a vector bundle over $\mathbb{C}^n \times D$ is trivial.

On the other hand, we can try to "algebraize" the previous problem: let C be the category of free modules over $\mathbb{C}\left[x, \frac{1}{x}\right]$, endowed with a derivation ∂ which is regular at infinity; let Ψ be the functor $(C) \to$ (vector bundles with connection over K) which associates to every E in C the vector bundle $E \otimes_{\mathbb{C}[x,\frac{1}{x}]} K$. We see that Ψ is an equivalence of categories as follows; first the fact that it is fully faithful results from the fact that an analytic horizontal section of $\hom_K(\Psi(E), \Psi(F))$ near zero extends to a meromorphic function at infinity (because of the hypothesis "regular singularities"). The surjectivity is then proved in the usual way (extend an (E,∂) over K to $\mathbb{C}^*$ by extending the local system of solutions, and compensate at infinity by a regular singularity).

We could then pose an algebraic moduli problem analogous to the previous one for the families of (C) formally isomorphic at the origin to the constant family (E,∂); I will not give a precise statement, because the following example shows that there exists no algebraic structure on $C\ell(E,\partial)_{an}$ that makes it a (even coarse) moduli space for these families.

We take $(E(\beta,\gamma),\partial) = (K^2,\partial)$ the family depending on $(\beta,\gamma) \in \mathbb{C}^2$ whose connection matrix is $\frac{1}{x^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{x}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ (α, δ fixed); for given β, γ, there exists a unique formal isomorphism given by $S = \mathrm{Id} +$(terms of order ≥ 1) near $(E(0,0),\partial)$.

On the other hand, $C\ell(E(0,0),\partial)_{an} = \mathbb{C}^2$; so we have an analytic map $\mathbb{C}^2 \to \mathbb{C}^2$ given by $(\beta,\gamma) \mapsto$ the class of $(E(\beta,\gamma),\partial)$. It turns out that we can here explicitly calculate this map [JLP76], [MR82]; among other things, we find that in general the inverse image of a point of $\mathbb{C}^2$ is countable; so this map is not "algebraizable".

A. Appendix

Proof of Theorem 3.5. Let $\mathcal{A}'$ be the subspace of $\mathcal{A}$ consisting of f whose asymptotic expansion has no poles; let $G\ell^0(n,\mathcal{A}')$ be the subsheaf of group of $G\ell(n,\mathcal{A}')$ formed of matrices asymptotic to the identity. By taking a basis $e_1,\cdots,e_n$ of E, it obviously suffices to prove the following assertion.

A.1 Proposition. The map $H^1(S,G\ell^0(n,\mathcal{A}')) \to H^1(S,G\ell(n,\mathcal{A}')$ has image 0.

We will first trivialize the situation "over C^∞"; let $\Gamma^{\mathbb{R}}$ be the sheaf on S defined as follows: the elements of $\Gamma^{\mathbb{R}}_\theta$ are represented by matrices of size n in a small closed sector $\{|\arg x-\theta| \le \varepsilon, |r| \le \varepsilon\} \cup \{0\}$ of the form $\mathrm{Id}+M$, M being C^∞ (with respect to Re x and Im x) and flat at 0, i.e. all its derivatives with respect to Re x and Im x are zero at 0; this is again a sheaf of groups over $\mathbb{R}$.

A.2 Lemma. $H^1(S,\Gamma^{\mathbb{R}}) = 0$.

Consider the "polar coordinates" map $S\times\mathbb{R}_+ \to \mathbb{C}$ defined by $(\theta,r)\mapsto re^{i\theta}$; by the inverse image of this map, $\Gamma_{\mathbb{R}}$ becomes the restriction to S of the sheaf of C^∞ matrices on $S\times\mathbb{R}_+$, tangent to the identity to infinite order along S. The elements of $H^1(S,\Gamma^{\mathbb{R}})$ therefore classify the C^∞ vector bundles over $S\times\mathbb{R}_+$ near S, formally trivial along S. Such a trivialization extends to E, which gives the desired result.

Next, take an open cover $\{U_i\}$ of S, and let $\{\beta_{ij}\}$ be a cocycle of $G\ell^0(n,\mathcal{A}')$ in this cover; the previous lemma shows that there exist $\alpha_i \in \Gamma(U_i,\Gamma^{\mathbb{R}})$ such that $\beta_{ij} = \alpha_i\alpha_j^{-1}$, or again $\alpha_i = \beta_{ij}\alpha_j$, whence $\alpha_i^{-1}\frac{\partial}{\partial\bar{x}}\alpha_i = \alpha_j^{-1}\frac{\partial}{\partial\bar{x}}\alpha_j$; let γ be the common value of these expressions; it is a C^∞ matrix near 0, and zero to infinite order at 0.

It is then well known that there exists a C^∞ matrix δ near 0, with $\delta(0) = \mathrm{Id}$, and such that $\delta^{-1}\frac{\partial\delta}{\partial\bar{x}} = \gamma$; set $\alpha'_i = \alpha_i\delta^{-1}$; the α'_i are holomorphic, $\alpha'_i(0) = \mathrm{Id}$, and $\beta_{ij} = \alpha'_i\alpha'^{-1}_j$. Whence the proposition. □

References

[Bir13] George D Birkhoff. The generalized riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations. In *Proceedings of the American Academy of Arts and Sciences*, volume 49, pages 521–568. JSTOR, 1913.

[BJL79] W. Balser, W. B. Jurkat, and D. A. Lutz. Birkhoff invariants and Stokes' multipliers for meromorphic linear differential equations. *J. Math. Anal. Appl.*, 71(1):48–94, 1979.

[Del] Deligne. lettres à b. malgrange. (8.1977 et 4.1978).

[JLP76] W. Jurkat, D. Lutz, and A. Peyerimhoff. Birkhoff invariants and effective calcualtions for meromorphic linear differential equations. *J. Math. Anal. Appl.*, 53(2):438–470, 1976.

[Jur78] W. B. Jurkat. *Meromorphe Differentialgleichungen*, volume 637 of *Lecture Notes in Mathematics*. Springer, Berlin, 1978.

[Lev75] A. H. M. Levelt. Jordan decomposition for a class of singular differential operators. *Ark. Mat.*, 13:1–27, 1975.

[Mal72] B. Malgrange. Sur les points singuliers des équations différentielles. In *Séminaire Goulaouic-Schwartz 1971–1972*, pages Exp. Nos. 20–22, 36. École Polytech., Paris, 1972. Équations aux dérivées partielles et analyse fonctionnelle,.

[Mal79] B. Malgrange. Remarques sur les équations différentielles à points singuliers irréguliers. In *Équations différentielles et systèmes de Pfaff dans le champ complexe (Sem., Inst. Rech. Math. Avancée, Strasbourg, 1975)*, volume 712 of *Lecture Notes in Math.*, pages 77–86. Springer, Berlin, 1979.

[MR82] Jean Martinet and Jean-Pierre Ramis. Problèmes de modules pour des équations différentielles non linéaires du premier ordre. *Inst. Hautes Études Sci. Publ. Math.*, (55):63–164, 1982.

[Rob80] P. Robba. Lemmes de Hensel pour les opérateurs différentiels. Application à la réduction formelle des équations différentielles. *Enseign. Math. (2)*, 26(3-4):279–311 (1981), 1980.

[Sib68] Yasutaka Sibuya. Perturbation of linear ordinary differential equations at irregular singular points. *Funkcial. Ekvac.*, 11:235–246, 1968.

[Sib77] Yasutaka Sibuya. Stokes phenomena. *Bull. Amer. Math. Soc.*, 83(5):1075–1077, 1977.

[Sib90] Yasutaka Sibuya. *Linear differential equations in the complex domain: problems of analytic continuation*, volume 82 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1990. Translated from the Japanese by the author.

[Was87] Wolfgang Wasow. *Asymptotic expansions for ordinary differential equations*. Dover Publications, Inc., New York, 1987. Reprint of the 1976 edition.