Combinatorics and Topology of Hyperplane Arrangements

Si-Yang Liu

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1. Combinatorics

(1a) Hyperplane Arrangements.

1.1 Definition. A **hyperplane arrangement** is a pair $\mathcal{V} = (V, \eta)$ where $V \subseteq \mathbb{R}^n$ is a linear subspace of dimension k transverse to all coordinate hyperplanes $H_i = \{x_i = 0\}$ and $\eta \in \mathbb{R}^N/V$ a vector.

We should regard the intersection $H_i \cap (V + \eta)$ to be affine hyperplanes in the Euclidean space $V + \eta \cong \mathbb{R}^n$, so we see that this formulation is the same as putting n affine hyperplanes in V.

1.2 Example. *Let $V \subset \mathbb{R}^n$ be a codimension 1 hyperplane defined by the equation*

$$V = \{x_1 + x_2 + \cdots + x_n = 0\}$$

and write $\eta = (1/n, 1/n, \cdots, 1/n)$. Then we obtain a hyperplane arrangement $\mathcal{V} = (V, \eta)$. Choosing an isomorphism $V + \eta \cong \mathbb{R}^{n-1}$ given by $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-1})$, we can write down the equations to all the hyperplanes as $H_i \cap (V + \eta) = \{x_i = 0\}$ for $1 \le i \le n-1$ and $H_n \cap (V + \eta) = \{x_1 + x_2 + \cdots + x_{n-1} + 1 = 0\}$. The two-dimensional picture is given below:

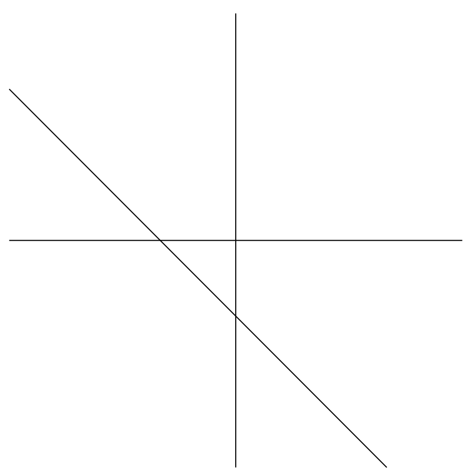

Figure 1: Two-dimensional Pair-of-pants

1.3 Example. *If $\dim V = 1$, then all intersections of H_i with $V + \eta$ will be points on a real line, and we can simply represent them as real numbers $a_1 < a_2 < \cdots < a_n$ in $\mathbb{R}$. For example, when $n = 3$, the picture looks like*

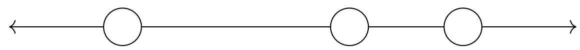

FIGURE 2: A 1-DIMENSIONAL HYPERPLANE ARRANGEMENT

An interesting question in combinatorics for hyperplane arrangements is to count the number of chambers divided by these hyperplanes. Equip $\mathbb{R}^n$ with its usual Euclidean topology, a **chamber** for a given hyperplane arrangement $\mathcal{V}$ is defined to be a connected component of the complement $V \setminus \bigcup_{i=1}^{n} H_i$. From this definition we directly see that the number of chambers in picture (1) is 7, while in (2) it is 4.

1.4 Definition. A chamber R is called **compact** if its closure $\bar{R}$ is compact in $V + \eta$.

For the above examples, the number of compact chambers is 1 in (1) and is 2 in (2). In fact, as we'll see, the numbers of compact chambers in all the examples of (1.2) are always 1. As an exercise, we can also easily show that

1.5 Proposition. If $\dim \mathcal{V} = 1$, then the number of chambers of $\mathcal{V}$ is $n+1$ and the number of compact chambers of $\mathcal{V}$ is $n-1$.

(1b) Delection-restriction induction. To detect the problem of counting chambers, note that it's always easy to count when the dimension of $\mathcal{V}$ and the number of hyperplanes are small, so we can try to attack the problem via induction.

1.6 Definition ([LLM20])**.** Let $\mathcal{V} = (V, \eta)$ be a hyperplane arrangement and H_i be one of the coordiante hyperplanes, then we write $\mathcal{V}_{H_i}$ for the hyperplane arrangement obtained by "deleting H_i", i.e. $\mathcal{V}_{H_i} = (W, \eta')$ where W is the image of V under the projection $\pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1}, (a_1, a_2, \cdots a_n) \mapsto (a_1, a_2, \cdots a_{i-1}, a_{i+1}, \cdots a_n)$, and $\eta' = \pi_i(\eta)$. Write $\mathcal{V}^{H_i}$ to be the hyperplane arrangement obtained by "restricting to H_i", i.e. $\mathcal{V}^{H_i} = (V \cap H_i, \eta')$ where $\eta' \in H_i/(H_i \cap V) \cong \mathbb{R}^n/V$ is the restriction of η to the subspace.

To count the number of chambers in a given hyperplane arrangement $\mathcal{V}$, note that if $\dim V = 1$, then the number has been determined by Proposition 1.5. In general, we can regard $\mathcal{V}$ as obtained by adding hyperplanes $H_1, \cdots H_n$ one-by-one, and we can look at the change to numbers when we add a single hyperplane. Without loss of generality, let's look at H_n: adding a single hyperplane divides each chamber it passes through into two pieces, so the number of chambers we have in $\mathcal{V}$ is the same as the number of chambers of $\mathcal{V}_{H_n}$ plus the number of chambers that H_n goes through, which is the same as the number of chambers of $\mathcal{V}^{H_n}$. Therefore we obtained the following theorem:

1.7 Theorem. Let $\mathcal{V}$ be a hyperplane arrangement and H any hyperplane in it, then we have

$$R(\mathcal{V}) = R(\mathcal{V}_H) + R(\mathcal{V}^H)$$

where $R(\mathcal{V})$ is the number of chambers of $\mathcal{V}$. Similarly, for compact chambers, we have

$$B(\mathcal{V}) = \begin{cases} B(\mathcal{V}_H) + B(\mathcal{V}^H), & \text{if } \operatorname{rank} \mathcal{V} = \operatorname{rank} \mathcal{V}_H; \\ 0, & \text{otherwise.} \end{cases}$$

Where a **rank** of $\mathcal{V}$ is the dimension of the subspace spanned by normals to each hyperplane.

In good cases, this theorem will give us beautiful combinatorial formulae for the numbers $R(\mathcal{V})$. We mainly treat with two cases.

(1c) Two examples. There're two spcial classes of hyperplane arrangements that we can do the computation by hand. Generically, the configuration of hyperplanes in a given plane will not have multiple intersections or parallel lines. In general,

1.8 Definition. A hyperplane arrangement $\mathcal{V}$ is called **in general position** if for any collection I of hyperplanes in $\mathcal{V}$, we have

$$\operatorname{codim} \bigcap_{i \in I} H_i = |I|.$$

In this case, the hyperplane H_n would intersect every other hyperplanes once, so $\mathcal{V}^{H_n}$ is again a hyperplane arrangement in general position. Therefore, write $R(n, k)$ for the number of chambers in a hyperplane arrangement in general position, we will get an induction formula $R(n, k) = R(n-1, k-1) + R(n, k-1)$ with $R(1, k) = k+1$ and $R(n, 1) = 2$. Therefore, we conclude that

1.9 Theorem. $R(n, k) = 1 + k + \binom{k}{2} + \cdots + \binom{k}{n}$.

Another example is when $\eta = 0$, which is very singular since all hyperplanes intersect at a point. Equivalently,

1.10 Definition. We say a hyperplane arrangement $\mathcal{V}$ is **central** if $\eta = 0$, or equivalently, there is a point $p \in V$ which lies in every hyperplane.

For the latter case, we readily have

1.11 Proposition. If $\mathcal{V}$ is central, then $B(\mathcal{V}) = 0$ and $R(\mathcal{V}) = 2n$.

Central hyperplane arrangements arose naturally in the study of representation theory: an abstract root system is a central hyperplane arrangement, where it admits an action of the Coxeter group by reflections. We call these examples **Coxeter arrangements**. See [FR05] for more information.

(1d) Intersection poset and characteristic polynomial. If we want the information for compact chambers, we need more combinatorial information subtracted from hyperplane arrangements. The information is encoded in a poset called intersection poset:

1.12 Definition. Let $\mathcal{V}$ be a hyperplane arrangement, then we associate to $\mathcal{V}$ a **lattice** $L(\mathcal{V})$ defined as follows. As a set,

$$L(\mathcal{V}) = \left\{ I \subseteq \{1, 2, \cdots, n\} \middle| \bigcap_{i \in I} H_i \neq \varnothing \right\},$$

and the partial order is given by inclusion. This lattice is called the **intersection poset** of $\mathcal{V}$.

The empty set - corresponding to the whole space V - is the unique minimum in $L(\mathcal{V})$. The intersection poset is always finite, and so we can represent an intersection poset as a graph, called **Hesse diagram**, by drawing a vertex for each element of $L(\mathcal{V})$, and an edge connecting vertices if they're **adjacent**, i.e. if $I < J$ but there're no sets K with $I < K < J$.

For each intersection poset $L(\mathcal{V})$, we associate a **Möbius function** $\mu : L(\mathcal{V}) \times L(\mathcal{V}) \to \mathbb{Z}$ as follows: $\mu(x, x) = 1$ for all $x \in L(\mathcal{V})$, and if $x < y$,

$$\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z).$$

This inductive formula will finally gives you the value of all points in $L(\mathcal{V})$. Since $L(\mathcal{V})$ has a prescribed minimum $\varnothing$, we get a function $\mu : L(\mathcal{V}) \to \mathbb{Z}$ defined by $\mu(x) := \mu(\varnothing, x)$. For each $x \in L(\mathcal{V})$, the **length** $\ell(x)$ of x is the minimal number of elements in a chain from $\varnothing$ to x.

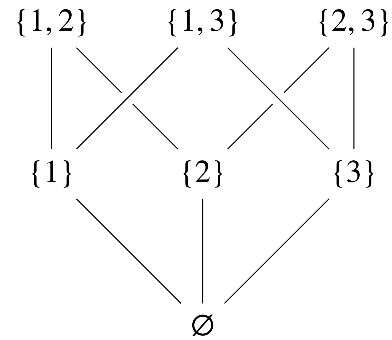

Figure 3: Hesse Diagram for a 2-dimensional Hyperplane Arrangement

1.13 Example. *In the above Hesse graph, we can write down the Möbius function via its values at all points:* $\mu(\varnothing) = 1$*,* $\mu(x) = -1$ *for all* x *with* $\ell(x) = 1$*, and* $\mu(y) = 1$ *for* $\mu(y) = 2$*.*

Let $f, g : L(\mathcal{V}) \times L(\mathcal{V}) \to \mathbb{Z}$ be functions, then we define their **convlution** to be

$$f * g(x, y) = \sum_{x \le z \le y} f(x, z) g(z, y),$$

then we have the following Möbius inversion formula:

1.14 Proposition ([S+04])**.** The following two conditions are equivalent for functions $f, g : L(\mathcal{V}) \to \mathbb{R}$:

$$f(x) = \sum_{y \ge x} g(y), \text{ for all } x \in L(\mathcal{V});$$
$$g(x) = \sum_{y \ge x} \mu(x, y) f(y), \text{ for all } x \in L(\mathcal{V}).$$

1.15 Definition. We define the **characteristic polynomial** $\chi_{\mathcal{V}}(t)$ of $\mathcal{V}$ to be

$$\chi_{\mathcal{V}}(t) := \sum_{x} \mu(x) t^{\dim(x)},$$

where $\dim x$ is the dimension of x viewed as intersection of hyperplanes.

We also have a deletion-restriction formula for characteristic polynomials:

1.16 Proposition. Let $H \in \mathcal{A}$ be a hyperplane in the hyperplane arrangement $\mathcal{A}$, then we have

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}_H}(t) - \chi_{\mathcal{A}^H}(t).$$

We will not give a combinatorial proof of this fact. In the subsequent section we will see the characteristic polynomial is the Poincaré polynomial of some space, and this result follows directly from an exact sequence of cohomology groups. Note that a hyperplane arrangement gives V a cellular decomposition, where the k-dimensional cells are exactly connected components of transverse intersections of k hyperplanes removing other transversely-intersecting hyperplanes. Therefore, if we look at the (modified) Euler characteristic of $\mathbb{R}^n$, we will get

$$(-1)^n = \chi(\mathbb{R}^n) = \sum_{x \in L(\mathcal{V})} (-1)^{\dim x} R(\mathcal{V}_x),$$

and by Möbius inversion formula, we conclude that

1.17 Proposition. $(-1)^n R(\mathcal{V}) = \chi_{\mathcal{V}}(-1)$.

A smilar analysis on $B(\mathcal{V})$ will show that

1.18 Proposition. $B(\mathcal{V}) = (-1)^{\operatorname{rank} \mathcal{V}} \chi_{\mathcal{V}}(1)$.

Therefore by directly computing the characteristic polynomial, we can obtain the numbers by plugging in values. For example,

1.19 Proposition ([S+04])**.** Let $\mathcal{V}$ be an n-dimensional hyperplane arrangement of k hyperplanes in general position, then

$$\chi_{\mathcal{V}}(t) = t^n - mt^{n-1} + \binom{k}{2} t^{n-2} - \cdots + (-1)^n \binom{k}{n}.$$

In particular, we have

$$R(\mathcal{V}) = 1 + k + \binom{k}{2} + \cdots + \binom{k}{n} \quad \text{and} \quad B(\mathcal{V}) = (-1)^n \left(1 - k + \binom{k}{2} - \cdots + (-1)^n \binom{k}{n}\right) = \binom{k-1}{n}.$$

The computation is an easy corollary of the following Whitney's theorem:

1.20 Theorem (Whitney)**.** Let $\mathcal{V}$ be an arrangement in an n-dimensional vector space, then

$$\chi_{\mathcal{V}}(t) = \sum_{\substack{\mathcal{U} \subseteq \mathcal{V} \\ \mathcal{U} \text{ is central}}} (-1)^{\#\mathcal{U}} t^{n - \operatorname{rank} \mathcal{U}}.$$

2. Topology

Now we discuss a relevant topology question that involves the combinatorics we have discussed above.

(2a) Motivation. In this section, we work with complex space $\mathbb{C}^d$ or the projective space $\mathbb{C}P^d$. A hyperplane arrangement in $\mathbb{C}^d$ would consist of a finite collection of complex hyperplanes in $\mathbb{C}^d$, which has complex codimension 1 and hence the complement

$$\mathbb{C}^d \setminus \bigcup_{1 \le i \le n} H_i$$

is connected. It makes no sense now to ask whether the complement is connected or not, but we are interested in the topology of this complement. Write $M(\mathcal{V})$ to be $\mathbb{C}^d$ removing these hyperplanes, and we want to compute the cohomology ring $H^*(M(\mathcal{V}))$. A natural idea, as suggested by Arnold, is that the deRham cohomology ring $H^*_{dR}(M(\mathcal{V}))$ is actually generated by differential 1-forms $\dfrac{\mathrm{d}L_{H_i}}{L_{H_i}}$, where L_{H_i} is the linear form with H_i as its zero set. This heuristic shows that the cohomology group of $H^*_{dR}(M(\mathcal{V}))$ should have some connection with the combinatorics of the hyperplane arrangement $\mathcal{V}$. The work of Orlik and Solomon [OS80] showed that the cohomology algebra actually comes from the intersection poset: we can associate to the intersection poset $L(\mathcal{V})$ an algebra $\mathscr{A}(\mathcal{V})$ using purely information from $L(\mathcal{V})$, and this algebra would then isomorphic to $H^*_{dR}(M(\mathcal{V}))$, and therefore we can read off all cohomological information from the intersection poset.

(2b) Computing Cohomology. In this paragraph we try to prove Arnold's heuristic, that is, to prove that the singular cohomology algebra $H^* M(\mathcal{V})$ is generated by the differential 1-forms $\dfrac{\mathrm{d}L_H}{L_H}$ for all $H \in \mathcal{V}$. The proof is a topological translation of deletion-restriction induction: we look at the triple $(\mathcal{V}, \mathcal{V}_H, \mathcal{V}^H)$ where $H \in \mathcal{V}$ is some hyperplane. Note that $M(\mathcal{V}) \hookrightarrow M(\mathcal{V}_H) \hookleftarrow M(\mathcal{V}^H)$, and this relation gives

2.1 Lemma. *There exists a short exact sequence of cohomology groups*

$$0 \to H^{k+1}(M(\mathcal{V})) \to H^{k+1}(M(\mathcal{V}_H)) \to H^k(M(\mathcal{V}^H)) \to 0.$$

Proof. For each hyperplane H, $M(\mathcal{V}_H)$ has an open neighbourhood in $M(\mathcal{V}_H)$ of the form $M(\mathcal{V}_H)\times\mathbb{D}$, hence the "deletion" process can be regarded as writing $M(\mathcal{V}_H)$ as a union

$$M(\mathcal{V}_H) = M(\mathcal{V}^H)\times\mathbb{D}\cup M(\mathcal{V})$$

with intersection isomorphic to $M(\mathcal{V}_H)\times\mathbb{C}^*$. Therefore we get a Mayer-Vietoris sequence

$$\cdots \to H^k M(\mathcal{V}_H) \to H^k M(\mathcal{V}) \oplus H^k(M(\mathcal{V}^H)\times\mathbb{D}) \to H^k(M(\mathcal{V}^H)\times\mathbb{D}^*) \to \cdots$$

Since $M(\mathcal{V}^H)\times\mathbb{D} \cong M(\mathcal{V}^H)$ and the cohomology group $H^k(M(\mathcal{V}^H\times\mathbb{D}^*)) \cong H^k(M(\mathcal{V}^H))\oplus H^{k-1}(M(\mathcal{V}^H))$ as cohomology groups, we get

$$H^*M(\mathcal{V}_H) \to H^*M(\mathcal{V})\oplus H^*M(\mathcal{V}^H) \to H^*M(\mathcal{V}^H)\oplus H^*M(\mathcal{V}^H)[-1] \xrightarrow{+1}$$

Note that the second map in this sequence is isomorphism onto $H^*M(\mathcal{V}^H)$ when restricting to $H^*M(\mathcal{V}^H)$, hence we can simply the sequence to be

$$H^*M(\mathcal{V}_H) \to H^*M(\mathcal{V}) \to H^*M(\mathcal{V}^H)[-1] \xrightarrow{+1}.$$

A careful look at the connecting homomorphism would tell us this map is zero, and hence we obtain the required short exact sequence. □

Now to show the generation of cohomology algebra, it suffices for us to prove the case in dimension 1, but in dimension 1 the result follows from a direct computation as in [BT$^+$82]. Therefore we conclude that the singular cohomology group of $M(\mathcal{V})$ is generated by those log differentials. Because of this, we also call the deRham cohomology of $M(\mathcal{V})$ **log cohomology**, written as $H^*_{log}M(\mathcal{V})$.

(2c) Orlik-Solomon algebra. Orlik and Solomon further showed that the log cohomology ring actually comes from the intersection poset. Let $\mathcal{V}$ be any complex hyperplane arrangement and let $L(\mathcal{V})$ be the corresponding intersection poset, then we can associate an algebra $\mathscr{B}(\mathcal{V})$ as follows: consider the exterior algebra $\mathcal{E} := \mathrm{Ext}^*_{\mathbb{Z}}[x|\ell(x)=1]$ generated by all elements of $L(\mathcal{V})$ with length 1 and degree 1. For $S=(a_1,\cdots,a_p)\in\{1,2,\cdots,n\}^p$, write e_S for the element $\bigwedge_{i=1}^{p} e_{a_i}$, where e_{a_i} denotes the length 1 element corresponding to the index a_i, then we know $\mathcal{E}$ is generated as a vector space by elements e_S for $S\in\{1,2,\cdots,n\}^p$ for all $0\le p\le n$(with $e_\varnothing = 1$). This algebra admits a natural differential $\partial : \mathcal{E}\to\mathcal{E}$ defined by

$$\partial e_S = \sum_{i=1}^{p}(-1)^{i-1} e_{a_1}\wedge\cdots\wedge\hat{e}_{a_i}\wedge\cdots\wedge e_{a_p},$$

where $S=\{a_1,a_2,\cdots,a_p\}$, with $\partial 1 = 0$ and $\partial e_a = 1$ for all $a\in\{1,2,\cdots,n\}$. We can verify that $\partial^2=0$ and for two elements S,T we have the Leibnitz law

$$\partial(e_S\wedge e_T) = (\partial e_S)e_T + (-1)^p e_S(\partial e_T)$$

where $p=|S|$. We say S is **independent** if the maximal element $\bigvee_{i=1}^{p} e_{a_i}$ has length p, and **dependent** if its length is less than p. Let $\mathcal{I}$ be the ideal generated by elements of the form ∂e_S for S dependent, and write $\mathscr{A} = \mathcal{E}/\mathcal{I}$ to be the quotient algebra of $\mathcal{E}$ by the ideal $\mathcal{I}$.

2.2 Theorem (Orlik-Solomon)**.** The Poincaré polynomial $P_{\mathscr{A}}(t)$ of $\mathscr{A}$ is

$$P_{\mathscr{A}}(t) = \sum_{x \in L(\mathcal{V})} \mu(x)(-t)^{\ell(x)}.$$

Note that it looks almost the same as the characteristic polynomial we defined in the previous section. Moreover, Orlik and Solomon proved that

2.3 Theorem (Orlik-Solomon)**.** Let $\mathcal{V}$ be a hyperplane arrangement of k complex hyperplanes in a complex space of dimension n, then there exists an isomorphism $\mathscr{A} \cong H^*_{log}(M(\mathcal{V}); \mathbb{Z})$ by sending e_S to $[\omega_S]$ for all S. In particular, the Poincaré polynomial of $M(\mathcal{V})$ is given by

$$P_{M(\mathcal{V})}(t) = \sum_{x \in L(\mathcal{V})} \mu(x)(-t)^{\ell(x)}.$$

Therefore the cohomology algebra of $M(\mathcal{V})$ is combinatorial in nature.

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