From Lie Groups to Group Algebras

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1. A Quick Review

(1a) Representative Functions. Let G be a Lie group and K either $\mathbb{R}$ or $\mathbb{C}$, then $C^0(G,K)$, the space of continuous functions from G to K admits two natural G-actions: the left action $L: G \times C^0(G,K) \to C^0(G,K);\ (g,f) \mapsto (h \mapsto f(gh))$ and the right action $R: C^0(G,K) \times G \to C^0(G,K);\ (f,g) \mapsto (h \mapsto f(hg^{-1}))$. Therefore we get two G-modules $(C^0(G,K),L)$ and $(C^0(G,K),R)$.

1.1 Definition. A function $f \in C^0(G,K)$ is called **representative** if the smallest G-module of $(C^0(G,K),R)$ containing f is finite-dimensional.

Let $f,g \in C^0(G,K)$ be representative functions, then it's easy to verify that fg and $f+g$ are also representative. Therefore the set $\mathscr{T}(G,K)$ of all representative functions is a subalgebra of $C^0(G,K)$. Since G is compact, it's also a subalgebra of $L^2(G,K)$, the ring of all L^2-functions with the unique integration induced from topological structure of G. Write $\mathrm{Irr}(G,K)$ to be the set of all irreducible K-representations of G, the Peter-Weyl theorem states that

1.2 Theorem (Peter-Weyl). (1) $\mathscr{T}(G,K)$ is dense in $C^0(G,K)$ and $L^2(G,K)$ with the respective topologies;

(2) The irreducible characters generate a dense subspace of the space of continuous class functions.

Given a (G,K)-representation V, by picking a basis we can write this representation as a matrix-valued function $(r_{ij})_{1\le i,j\le \dim V}$. The functions r_{ij} generate a subspace of $C^0(G,K)$ invariant under the $G\times G$-action on both the left and the right. Therefore we get a $G\times G$-submodule of $C^0(G,K)$. Note that matrices can be regarded as tensor products $V^* \times V$ with induced G-actions, hence we can regard the above assignment as a G-equivariant map $V^* \otimes V \to C^0(G,K)$, mapping $\varphi \otimes v \in V^* \otimes V$ to the function $g \mapsto \varphi(gv)$.

1.3 Proposition. Any G-submodule B of $C^0(G,K)$ is a direct sum of submodules $B \cap (V^* \otimes V)$ for V irreducible.

1.4 Proposition. We have a direct sum decomposition of $G \times G$-modules

$$\mathscr{T}(G,K) \cong \bigoplus_{V \text{ irreducible}} V^* \otimes V.$$

Here by irreducible we mean finite-dimensional G-modules which are irreducible.

(1b) Characters. We regard $C^0(G,K)$ not only as a $G\times G$-module, but an algebra: for $\varphi, \psi \in C^0(G,K)$, define the product $\varphi * \psi$ by

$$\varphi * \psi(g) = \int_G \varphi(gh^{-1})\psi(h)\mathrm{d}h.$$

This gives $C^0(G,K)$ a Banach algebra structure. A little bit more analysis tells us that $L^2(G,K)$ with this convolution product also gives a Banach algebra structure on $L^2(G,K)$. Note that by Peter-Weyl theorem,

$\mathscr{T}(G,K)$ is dense in $C^0(G,K)$ and also $L^2(G,K)$, so to study the algebraic structure of $\mathscr{A}(G,K)$(either the algebra of continuous or L^2 functions), we can start with these representative functions.

1.5 Definition. Let V be a (G,K)-representation. The **character** $\chi_V : G \to K$ of V is given by $\chi_V(g) = \mathrm{Tr}(g)$, where $\mathrm{Tr} : G \to K$ is the trace function, i.e. pick any basis of V, representing each g as matrices, and take the trace of these matrices.

Note that $\chi_V \in V^* \otimes V$ since once we pick a basis $e_1, \cdots, e_n$ of V and the corresponding dual basis $e^1, \cdots, e^n \in V^*$, the function is given by $\chi_V = \sum_{i=1}^{n} e^i \otimes e_i$. Therefore via the embedding we have $\chi_V \in \mathscr{A}(G,K)$.

1.6 Proposition. Let V and W be two non-isomorphic irreducible G-modules and (u_{ij}), (v_{ij}) matrix representations of U and V, respectively, then the characters $\chi_U,\ \chi_V$ satisfies the following:

(i) $\int_G |\chi_U|^2 \mathrm{d}g = \int_G |\chi_V|^2 \mathrm{d}g = 1$ and $\int_G \chi_U \bar{\chi}_V \mathrm{d}g = 0$.

(ii) $\varphi * \psi = \psi * \varphi$ for all functions $\varphi,\ \psi \in \mathscr{A}(G,K)$, hence the algebra $\mathscr{A}(G,K)$ is commutative.

(iii) If U and V are both unitary, then $u_{ij} * u_{kl} = \delta_{jk} u_{il}$ where δ_{jk} is the Kronecker delta function, and $u_{ij} * v_{kl} = 0$ for any i,j,k,l chosen.

(iv) $\chi_U * u_{ij} = u_{ij}$, $\chi_V * v_{ij} = v_{ij}$ for all i,j.

2. Group Algebra From Lie Groups

Given a finite group G, we can associate a **group algebra** $K[G] := \{\varphi : G \to K\}$ with additions given by $(\varphi, \psi) \mapsto \varphi + \psi$ and products given by convolution products

$$\varphi * \psi(g) = \int_G \varphi(gh^{-1})\psi(h)\mathrm{d}h.$$

Then we know that representations of G are the same as $K[G]$-modules. For a given compact Lie group G, we have two different models for group algebras: the Banach algebra $C^0(G,K)$ and the Hilbert algebra $L^2(G,K)$(i.e. an algebra which carries a complete bi-invariant inner product).

(2a) Banach group algebra. In this section we discuss mainly about Banach algebras and its modules. Write $\mathscr{B}$ as our Banach algebra $C^0(G,K)$.

2.1 Definition. A **$\mathscr{B}$-module** V is a locally convex topological vector space endowed with a continuous map $* : \mathscr{B} \times V \to V$ such that

- $(f * g) * v = f * (g * v)$;
- $1 * v = v$ where $1 \in \mathscr{B}$ is the unit.

We make some digression into analysis before we move on. For any Banach or Hilbert G-vector space V, we define $C^0(G,V)$ to be the set of all continuous maps from G to V, with compact-open topology.

2.2 Proposition. There exists a continuous linear map

$$\int_G : C^0(G,V) \to V, \quad f \mapsto \int_G f = \int_G f(g)\mathrm{d}g$$

satisfying the following properties:

(1) $\int v\mathrm{d}g = v$ for all $v \in V$.

(2) $\int f(g)\mathrm{d}g = \int f(g^{-1})\mathrm{d}g$.

(3) $p\left(\int_G f\right) \leq \int_G pf$ for each continuous seminorm $p: V \to \mathbb{R}$ and any $f \in C^0(G, V)$.

(4) If $L: V_1 \to V_2$ is a continuous linear map between locally convex complete Hausdorff spaces, then $L\int f = \int Lf$ for all $f \in C^0(G, V_1)$.

Using this invariant integral, we can then transfer any Banach G-representation V into a Banach $\mathscr{B}$-module as follows: for each $\varphi \in \mathscr{B}$ and $v \in V$, define

$$\varphi * v := \int_G \varphi(g)gv\mathrm{d}g.$$

We can verify that this gives a continuous $\mathscr{B}$-action on V, hence giving a $\mathscr{B}$-module. The converse will be discussed later. Note that $\mathscr{T}(G, K) \subseteq \mathscr{B}$ is dense in $\mathscr{B}$, and furthermore, we have

2.3 Proposition. If f is representative and $v \in V$, then $f * v$ is contained in a finite-dimensional G-subspace of V.

Proof. For each $g \in G$, we have

$$g(f * v) = g\int_G f(h)hv\mathrm{d}h = \int_G f(h)ghv\mathrm{d}h = \int_G f(g^{-1}h)hv\mathrm{d}h.$$

Note that $f(g^{-1}v)$ is in the G-orbit of f in $\mathscr{T}(G, K)$, which is contained in a finite G-subspace W of $\mathscr{T}(G, K)$, and the map $f \mapsto f * v$ for fixed $v \in V$ is a G-equivariant linear map $\mathscr{T}(G, K) \to V$, the image of W via this map is finite-dimensional, which is a finite-dimensional G-subspace containing $f * v$. □

As a Corollary, we have

2.4 Corollary. Let V_s be the direct sum of all finite-dimensional G-subspaces of V, then V_s is dense in V.

Proof. Since $\mathscr{T}(G, K)$ is dense in $\mathscr{B}$, the image of $\mathscr{T}(G, K) \times V \to V$ in V, written as W, is also dense. If U is a finite-dimensional G-subspace of V, then it can be decomposed into direct sum of irreducible G-subspaces, hence we can without loss of generality assume U is irreducible. Fix $u \in U$ and consider the G-subspace $\mathscr{T}(G, K)u \subset U$. Since $\mathscr{B}$ acts surjectively on U, $\mathscr{T}(G, K)$ is dense and this action is continuous, the subspace is non-trivial, hence by Schur's lemma $\mathscr{T}(G, K)u = U$. This proves that every finite-dimensional G-subspace is in the image of $\mathscr{T}(G, K) \times V \to V$. Proposition 2.3 implies that W is a direct sum of finite-dimensional subspaces, hence $W = V_s$. □

In particular, we have

2.5 Corollary. If V is an irreducible Banach G-representation, then V is finite-dimensional.

(2b) Hilbert group algebra. Now we discuss some further structures on a Hilbert group algebra $L^2(G, \mathbb{C}) := \mathscr{H}$. Recall that given two functions φ and ψ, the inner product of φ and ψ is given by

$$\langle \varphi, \psi \rangle := \int_G \varphi\bar{\psi}\mathrm{d}g.$$

In this case, we only consider Hilbert a.k.a. unitary representations of $\mathscr{H}$, which means that any $\mathscr{H}$-module V carries an inner product $\langle -, - \rangle_V$ such that for all $\varphi \in \mathscr{H}$ and all $v, w \in V$ we have $\langle \varphi * v, w \rangle_V = \langle v, \varphi^* * w \rangle_V$. In this case, the algebra of representative functions $\mathscr{T}(G, \mathbb{C})$ is still dense in $\mathscr{H}$, and for each irreducible representation V, we know that

- $\chi_V \in V^* \otimes V$ is an idempotent of $\mathscr{T}(G, \mathbb{C})$.
- Each $V^* \otimes V$ is a subalgebra of $\mathscr{T}(G, \mathbb{C})$.

Therefore we have a decomposition of algebras

$$\mathscr{T}(G, \mathbb{C}) = \bigoplus_{V \in \operatorname{Irr}(G, \mathbb{C})} \chi_V * \mathscr{T}(G, \mathbb{C}) * \chi_V \cong \bigoplus_{V \in \operatorname{Irr}(G, \mathbb{C})} V^* \otimes V.$$

Let V be a unitary G-module, then the same construction gives us a Hilbert $\mathscr{H}$-module structure on V. As a Banach module, we know that V has a dense subspace V_s consisting of finite-dimensional G-representations. In the case of Hilbert modules, we can give a finer description. Let χ be an irreducible character, define an operator

$$P_\chi : V \to V$$

defined by $P_\chi(v) = \chi * v$.

2.6 Proposition. (1) P_χ is an orthogonal projector with image V_χ.

(2) V_χ and V_ψ are orthogonal if $\chi \neq \psi$.

(3) V is the Hilbert sum of the V_χ for $\chi \in \operatorname{Irr}(G, \mathbb{C})$.

(4) V_χ is the smallest closed subspace of V containing all irreducible subspaces with character χ.

Proof. Since $\chi * \chi = \chi$, P_χ is a projector. It's orthogonal since V is a Hilbert module. If $\chi \neq \psi$, pick $v \in V_\chi$ and $w \in V_\psi$, we have $\langle v, w\rangle_V = \langle \chi * v, \psi * w\rangle = \langle \psi^* * \chi * v, w\rangle_V = 0$ since $\psi^* \in \psi * \mathscr{T}(G, \mathbb{C}) * \psi$. The fourth statement also follows since $\psi * \chi = 0$ whenever ψ and χ are distinct. Finally, the direct sum of all these V_χ is exactly the G-subspace V_s of all finite-dimensional G-subspaces, hence V is the closure of the direct sum, which is the so-called Hilbert sum. $\square$

(2c) Induced representation. As an application, let's talk about construction of induced representations. Given a closed subgroup $H \subseteq G$, we have a natural restriction functor $\operatorname{res}_H^G : G\,\mathsf{Rep} \to H\,\mathsf{Rep}$ between category of representations sending each G-representation V to V regarded as an H-representation. The aim of this section is to construct an adjoint functor $\operatorname{ind}_H^G : H\,\mathsf{Rep} \to G\,\mathsf{Rep}$. Let the group algebra $\mathscr{A}(G, K)$ to be either the Banach group algebra or the Hilbert one, from the previous section we have the following diagram of functors:

$$\begin{tikzcd} G\,\mathsf{Rep} \arrow[r, "\operatorname{res}_H^G"] \arrow[d] & H\,\mathsf{Rep} \arrow[d] \\ \mathscr{A}(G)\,\mathsf{Rep} \arrow[r, "\operatorname{res}_H^G"] & \mathscr{A}(H)\,\mathsf{Rep} \end{tikzcd}$$

Where the bottom arrow is simply given by "forgetting the $\mathscr{A}(G)$-module structure". We know that the functor res_H^G is left adjoint to the functor $\operatorname{ind}_H^G : \mathscr{A}(H)\,\mathsf{Rep} \to \mathscr{A}(G)\,\mathsf{Rep}$ given by tensoring on the left by $\mathscr{A}(G)$, i.e. we have the following bijection

$$\hom_{\mathscr{A}(H)}(\operatorname{res}_H^G V, W) \xrightarrow{\simeq} \hom_{\mathscr{A}(G)}(V, \operatorname{ind}_H^G W).$$

However, we haven't discussed how to go back from group algebras to Lie groups yet, so we should only use the above discussion as a heuristic and construct the induced representation by hand. Let V be an H-representation(possibly infinite-dimensional), consider the space

$$\operatorname{ind}_H^G V := G \times_H V = \{(g, v) | (gh, v) = (g, hv)\ \forall\, h \in H\}.$$

$\operatorname{ind}_H^G V$ has a natural projection(into the first component) to the homogeneous space G/H, whose fiber at each point is the Banach space V. On the other hand, we can interpret the tensor product $\mathscr{A}(G) \otimes_{\mathscr{A}(H)} V$ as the space $\mathscr{A}(G,V)^H$ of functions $f : G \to V$ which is invariant underthe H-action. Now we have

2.7 Proposition. There is a bijection between H-invariant functions $f : G \to V$ and continuous sections of $G \times_H V \to G/H$.

Both of them are called the **induced representation** of V from H to G. Furthermore, we have the adjunction

2.8 Proposition. Let V be a G-representation and W an H-representation, then we have an isomorphism

$$\hom_G(V, \operatorname{ind}_H^G W) \xrightarrow{\cong} \hom_H(\operatorname{res}_H^G V, W).$$

This is called **Frobenius reciprocity**. This adjunction can be useful in computation: for instance, if V is a trivial one-dimensional H-representation, then $\operatorname{ind}_H^G V \cong \mathscr{A}(G/H, K)$. By Frobenius reciprocity, we know that $\hom_G(W, \operatorname{ind}_H^G V) \cong \hom_H(\operatorname{res}_H^G W, V) = \hom_H(\operatorname{res}_H^G W, K)$, therefore for each $W \in \operatorname{Irr}(G, K)$, the multiplicity of W in $\operatorname{ind}_H^G V$ is exactly the dimension $\dim W^H$ of the subspace of fixed points under the H-action.

3. Tannaka-Kreĭn Duality

In this section, we discuss how to go from a given group algebra back to a Lie group. This depends in particular some further algebraic structures on the group algebra, which makes it into a Hopf algebra. As an application, we'll discuss the complexification of compact Lie groups, and the relation of their representations.

(3a) From group algebras back to Lie groups. In this section, we want to reconstruct a Lie group G from its group algebra $C^0(G, \mathbb{R})$. Recall that given a compact Hausdorff space X, the Banach space $C^0(X, \mathbb{C})$ is a commutative unital C^*-algebra. Conversely, given any commutative unital C^*-algebra B, we can construct a corresponding compact Hausdorff space X as

$$X = \{\mathfrak{m} | \mathfrak{m} \text{ is a maximal ideal of } B\}.$$

We can verify that B is the algebra of continuous functions on X. In the Lie group case, we need some further structure to construct this correspondence. Note that a maximal ideal of a Banach K-algebra is the same as an algebra homomorphism $B \to K$, so we can say $X = \hom_{\mathsf{Alg}}(B, K)$. In our case, we consider the algebra of representative functions $\mathscr{T}(G, K)$ and define the space G_K as

$$G_K = \hom_{K\,\mathsf{Alg}}(\mathscr{T}, K).$$

To see that G_K has a group structure, we need further algebraic structures on $\mathscr{T}(G, K)$. First observe that

3.1 Proposition. Let G and H be two compact Lie groups, then we have an isomorphism

$$\mathscr{T}(G \times H, K) \cong \mathscr{T}(G, K) \otimes \mathscr{T}(H, K).$$

Proof. Let $f \in \mathscr{T}(G, K)$ and $g \in \mathscr{T}(H, K)$, then fg is representative in $C^0(G \times H, K)$, hence we get a map $\mathscr{T}(G, K) \otimes \mathscr{T}(H, K) \to \mathscr{T}(G \times H, K)$, which can be verified to be a homomorphism of algebras. Conversely, if u is representative for $G \times H$, then after choosing a basis, we can write it as a linear combination of pairs of representative functions for G and for H, hence giving a map from $\mathscr{T}(G \times H, K) \to \mathscr{T}(G, K) \otimes \mathscr{T}(H, K)$. Finally we can check these two maps are mutually inverse. □

Then given any algebra $\mathscr{T}(G,K)$, we have a natural diagonal map $\Delta : \mathscr{T}(G,K) \to \mathscr{T}(G\times G,K)$ by taking any representative function u to $u\otimes u$. This is the coproduct operation on $\mathscr{T}(G,K)$, and we also have a counit map $\epsilon : \mathscr{T}(G,K) \to K$ by sending any representative function u to the value $u(1)$ as well as a coinverse map $\eta : \mathscr{T} \to \mathscr{T}$ defined by $\eta\varphi(g) = \varphi(g^{-1})$. These operations satisfy the following relations:

- Δ is co-associative, i.e. for any $\varphi \in \mathscr{T}(G,K)$ we have $\Delta^2(\varphi)\otimes\Delta(\varphi) = \Delta(\varphi)\otimes\Delta^2(\varphi)$.
- $\epsilon(\varphi) = \epsilon(\Delta(\varphi))\Delta(\varphi) = \Delta(\varphi)\epsilon(\Delta(\varphi))$.
- $\eta\Delta(\varphi)\Delta(\varphi) = \epsilon(\varphi)$.

We call an algebra admitting a coproduct, counit and coinverse structure satisfying the above relations a **Hopf algebra**. Now we easily see that

3.2 Lemma. *G_K is a group.*

The remaining part of this section aims to prove that $G_K \cong G$ as topological(hence Lie) groups. Note that for each $g \in G$, we have a natural evaluation map

$$\mathrm{ev}_g : \mathscr{T}(G,K) \to K : \varphi \mapsto \varphi(g).$$

This map can be verified to be a homomorphism of algebras, hence $\mathrm{ev}_g \in G_K$, and we get a map $G \to G_K$.

3.3 Proposition. The map $G \to G_K$ is injective for all K.

Proof. If there exists an element $g \in G$ such that $\mathrm{ev}_g = \mathrm{ev}_1$, pick any $V \in \mathrm{Irr}(G,K)$, we have $\chi_V(g) = \dim V^g = \chi_V(1) = \dim V$ and hence g acts as identity on all irreducible representations of G. If $g \neq 1$, pick an abelian subgroup T containing g, and pick a one-dimensional non-trivial representation W of T, we get that $\chi_{\mathrm{ind}_T^G W}(g) \neq \dim \mathrm{ind}_T^G W$, a contradiction. Therefore we should have $g = 1$, and hence G injects into G_K. □

We define the topology on G_K as the weakest topology such that all evaluation maps $\mathrm{ev}_\varphi : G_K \to K$, $\mathrm{ev}_\varphi(f) = f(\varphi)$ are continuous, then we can verify that this makes G_K into a topological group. Moreover,

3.4 Lemma. *The injection $G \to G_K$ is continuous.*

Now we need to prove the converse result, i.e. the map $G \to G_\mathbb{R}$ is a homeomorphism, so that $G_\mathbb{R}$ completely recovers G. Firstly, if we have an embedding $r : G \to GL(n,K)$, then each function r_{ij} lies in $\mathscr{T}(G,K)$, and hence r induces a map $r_K : G_K \to GL(n,K)$ mapping each $f \in G_K$ to $(f(r_{ij}))$.

3.5 Proposition. (1) The following diagram

$$\begin{tikzcd} G \arrow[rr, "i"] \arrow[dr, "r"'] & & G_K \arrow[dl, "r_K"] \\ & GL(n,K) & \end{tikzcd}$$

is commutative, and if r generates $\mathscr{T}(G,K)$, then r_K is injective.

(2) If $K = \mathbb{R}$, and r is an orthogonal representation, then $r_\mathbb{R} G_\mathbb{R} \subseteq O(n,\mathbb{R})$ is a closed subgroup.

Proof. If r generates $\mathscr{T}(G,K)$, then for two algebraic homomorphisms $f,\ g$ so that $r_K(f) = r_K(g)$, we have $f(r_{ij}) = g(r_{ij})$. Now since (r_{ij}) generates $\mathscr{T}(G,K)$, we have $f = g$, and therefore r_K is injective. If $K = \mathbb{R}$ and r maps G into $O(n)$, then we should have $(r_{ij})^t(r_{ij}) = \mathrm{id}$, and since any $f \in G_\mathbb{R}$ is a homomorphism of algebras, it follows that $(f(r_{ij}))^t(f(r_{ij})) = \mathrm{id}$. This proves that $r_\mathbb{R}$ also maps $G_\mathbb{R}$ into $O(n)$. Finally, we need to show $G_\mathbb{R}$ is compact, so that its image is a closed subgroup of $O(n)$ and $r_{\mathbb{R}}$ is an embedding. Since (r_{ij}) generates $\mathcal{T}(G,\mathbb{R})$, any element of $\mathcal{T}(G,\mathbb{R})$ is of the form $P(r_{ij})$ where P is a polynomial in n^2 coefficients, and the (r_{ij}) is in $O(n)$, which implies that the image of the function P is a closed interval I_P in $\mathbb{R}$. Now the evaluation maps ev_P together gives a continuous injection

$$\prod_{P\in\mathcal{T}(G,\mathbb{R})} : G_{\mathbb{R}} \to \prod_{P\in\mathcal{T}(G,\mathbb{R})} I_P.$$

Since each I_P is compact, the product $\prod I_P$ is also compact. Since we go through all the evaluation maps, this product map is injective, and since each evaluation maps are continuous, the product map is continuous and can be seen to be an embedding. An element $(a_P) \in \prod I_P$ is in the image of $\prod \mathrm{ev}_P$ if and only if $a_{PQ} = a_P a_Q$, $a_1 = 1$ and $a_{rP} = ra_P$ for all $P, Q \in \mathcal{T}(G,\mathbb{R})$. This implies that the image of $\prod \mathrm{ev}_P$ is compact. Therefore $G_{\mathbb{R}}$ is compact and hence its image under $r_{\mathbb{R}}$ in $O(n)$ is a closed subgroup. □

With this result, we can finally prove that

3.6 Theorem. $i : G \to G_{\mathbb{R}}$ is an isomorphism of Lie groups.

Proof. It suffices for us to prove that i is surjective, which is equivalent to the fact that $i^* : \mathcal{T}(G_{\mathbb{R}},\mathbb{R}) \cong \mathcal{T}(G,\mathbb{R})$(since then i^* induce the isomorphism $C^0(G_{\mathbb{R}},\mathbb{R}) \cong C^0(G,\mathbb{R})$, hence by picking open neighbourhoods and constructing bump functions we know that i should be surjective). Pick an orthogonal representation of G so that $r : G \hookrightarrow O(n)$, then $r_{\mathbb{R}}$ is also an embedding. Combining with the above commutative diagram, we can conclude that i^* maps generators to generators, hence inducing an isomorphism on the algebra of representative functions. □

As a final remark, I should mention that we only give half of the correspondence between group algebras and Lie groups. The other direction goes as follows: given a Hopf algebra H_K over K, we can construct a group G_K, and we should verify that the algebra $\mathcal{T}(G_K, K)$ is again the algebra H_K. This takes too much time to explain so I only leave as a remark, and the details are omitted.

(3b) Complexification. Finally, we briefly discuss the complexification of G, which is exactly the group $G_{\mathbb{C}}$. If $r : G \to GL(n,\mathbb{C})$ is an inclusion, then by Proposition 3.5 we get an inclusion $r_{\mathbb{C}} : G_{\mathbb{C}} \to GL(n,\mathbb{C})$. We want to prove that $G_{\mathbb{C}}$ is a complex Lie group(i.e. it's holomorphic) by arguing that $r_{\mathbb{C}}G_{\mathbb{C}}$ is a closed subgroup of $GL(n,\mathbb{C})$. The faithful representation r gives us a surjective ring homomorphism $r^* : \mathbb{C}[X_{ij}] \to \mathcal{T}(G,\mathbb{C})$ by mapping X_{ij} to r_{ij}, hence we have an isomorphism

$$\mathcal{T}(G,\mathbb{C}) \cong \mathbb{C}[X_{ij}]/I,$$

I being the kernel of r^*. Consider the analytic set $V(I) \subset \mathbb{C}^{n^2}$ given by $\{x \in \mathbb{C}^{n^2} | p(x) = 0 \ \forall\ p \in \mathbb{C}[X_{ij}]\}$.

3.7 Lemma. *There is a natural bijection $\sigma : V(I) \to G_{\mathbb{C}}$ inverse to $r_{\mathbb{C}}$.*

This is "Hilbert's Nullstellensatz" from algebraic geometry. Equip $V(I)$ with the subspace topology, then $V(I)$ is naturally an analytic set, and it suffices for us to prove that the map $r_{\mathbb{C}}$ is a homeomorphism, so that $G_{\mathbb{C}}$ is naturally holomorphic. Note that the topology of $G_{\mathbb{C}}$ is defined by evaluation maps ev_p for any $p \in \mathcal{T}(G,\mathbb{C})$, but then $\mathcal{T}(G,\mathbb{C})$ is generated by the representative functions r_{ij}, so the topology is defined by the maps $\mathrm{ev}_{r_{ij}}$. From this description we know that σ is continuous, hence $r_{\mathbb{C}}$ is a homeomorphism, and therefore $G_{\mathbb{C}}$ is holomorphic.

Suppose r is now a unitary representation, i.e. the image of r is in $U(n)$. Recall that on $GL(n,\mathbb{C})$ we have the polar decomposition given by $A = HP$ for $H \in U(n)$ and $P \in P(n)$ is a positive-definite hermitian matrix. As a space, we get a decomposition $GL(n,\mathbb{C}) \cong U(n) \times P(n)$.

3.8 Proposition. (1) If $A \in G_{\mathbb{C}}$ and $A = HP$, then H and P both lie in $G_{\mathbb{C}}$, hence we have a decomposition as spaces $G_{\mathbb{C}} \cong (U(n) \cap G_{\mathbb{C}}) \times (P(n) \cap G_{\mathbb{C}})$.

(2) $G_{\mathbb{C}} \cap P(n)$ is homeomorphic to an Euclidean space of dimension $\dim G = \dim(G_{\mathbb{C}} \cap U(n))$.

(3) $G_{\mathbb{C}} \cap U(n)$ is a maximal compact subgroup of $G_{\mathbb{C}}$.

Proof. (1) If $A = HP$, then we can find $U \in U(n)$ such that $UPU^{-1} = D$ where D is a diagonal matrix with positive real entries, hence there exists a diagonal matrix Z with $D = \exp Z$. Let J be the ideal generated by $U(r_{ij})U^{-1}$, so that $V(J) = UV(I)U^{-1}$, and we want to show that $D \in V(J)$. Note that H is unitary, we have $A^*A = P^2 \in V(I)$, hence $P^{2k} \in V(J)$ and therefore $D^{2k} \in V(J)$ for all $k \in \mathbb{Z}$. Note that $D^{2k} = \exp(2kZ)$, for all $Q \in J$, we have $Q(\exp(2Z)) = 0$, then for all $t \in \mathbb{R}$, we have $Q(\exp(tZ)) = 0$, and in particular when $t = 1$, we get that $Q(D) = 0$, and therefore $D \in V(J)$ and hence $P \in G_{\mathbb{C}}$. $H = AP^{-1} \in G_{\mathbb{C}}$.

(2) Since $GL(n, \mathbb{C}) \cong U(n) \times P(n)$, we have the corresponding Lie algebra satisfies $\mathfrak{gl}_n(\mathbb{C}) \cong \mathfrak{u}(n) \oplus \mathfrak{p}(n)$, where $\mathfrak{u}(n)$ is the space of all skew-hermitian matrices $A^* + A = 0$, and $\mathfrak{p}(n)$ is the space of all hermitian matrices. Therefore we know that $\mathfrak{p}(n) = i\mathfrak{u}(n)$ and hence $\mathfrak{gl}_n(\mathbb{C}) \cong \mathfrak{u}(n) \otimes_{\mathbb{R}} \mathbb{C}$. Similarly, given $G_{\mathbb{C}}$, we want to prove that $G_{\mathbb{C}} \cap P(n)$ is homeomorphic via exponential map to the Lie algebra $iL(G_{\mathbb{C}} \cap U(n))$. Let $X \in L(G_{\mathbb{C}} \cap U(n))$, then clearly $\exp X \in U(n) \cap G_{\mathbb{C}}$ and $\exp(iX) \in P(n)$. Let $p \in I$ so that $p(\exp X) = 0$ for such X. Replacing $\exp X$ by $\exp(tX)$ for $t \in \mathbb{C}$ gives an entire function $p(\exp tX)$ on t, which has a zero for all $t \in \mathbb{R}$, hence it's identically zero for all t, which implies that $\exp(iX) \in G_{\mathbb{C}}$. Therefore $\exp(iX) \in G_{\mathbb{C}} \cap P(n)$ and hence $\exp(i-)$ maps $L(G_{\mathbb{C}} \cap U(n))$ into $G_{\mathbb{C}} \cap P(n)$.

On the other hand, for each $A \in P(n) \cap G_{\mathbb{C}}$, we know that $A = \exp iX$ for some $X \in U(n)$. The same argument shows that $\exp X \in G_{\mathbb{C}}$, hence $\exp(i-)$ is surjective. Note that exp is a local diffeomorphism, hence it's a diffeomorphism from $L(G_{\mathbb{C}} \cap U(n))$ to $G_{\mathbb{C}} \cap P(n)$.

(3) If K is a compact Lie group containing G as its proper subgroup, then there is an element $A \in K \subsetneq G_{\mathbb{C}}$ which is not in G. Write $A = UP$ for $U \in G$, then we know that $P \neq \mathrm{id}$ and $P \in K$ since $U \in G \subsetneq K$. Now $tP \in K$ for all $t \in \mathbb{R}_+$, which implies that K is not compact, a contradiction. Therefore $G \subsetneq G_{\mathbb{C}}$ is a maximal compact subgroup of $G_{\mathbb{C}}$. □

The second statement in the previous Proposition shows that $G_{\mathbb{C}}$ is actually a **complexification** of G. Moreover, we can relate their representations as follows:

3.9 Proposition. If V is a unitary representation of G with $r : G \to U(n)$, then the induced embedding $r_{\mathbb{C}} : G_{\mathbb{C}} \to GL(n, \mathbb{C})$ is holomorphic and hence V is a holomorphic representation of $G_{\mathbb{C}}$. More over, the map $r_{\mathbb{C}}$ is uniquely determined by r.

Proof. The reason is that $G_{\mathbb{C}} \cap P(n) \cong iL(G)$, hence the image of $G_{\mathbb{C}}$ in $GL(n, \mathbb{C})$ is uniquely determined. Holomorphicity is just tautology. □

Therefore the holomorphic representations of $G_{\mathbb{C}}$ is exactly the same as unitary representations of G, and in particular, irreducible representations of G is the same as irreducible holomorphic representations of $G_{\mathbb{C}}$. We also sometimes work in the algebro-geometric context, where we call $G_{\mathbb{C}}$ an **algebraic group**, and holomorphic representations of $G_{\mathbb{C}}$ **algebraic representations**.