Oct. 21^{st}

MATH 125.

Outline

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- Drawing graphs of functions
- Mean Value Theorem Again
- (Optional) Linear Approximation

Drawing Graphs of Functions.

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△ Well-known models : $y=\frac{1}{x}$, $y=\sin x$, $y=x^2+ax+b$, $a,b\in\mathbb{R}$, ⋯

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△ Elementary transformations: $f(x+c)$, $f(cx)$, $f(x)+c$, $cf(x)$,

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operations on graphs of functions

Now : derivative $\rightsquigarrow$ monotonicity & convexity.

Monotonicity and Convexity

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$y = f(x)$ differentiable on (a, b).

Monotonicity and Convexity

$y = f(x)$ differentiable in (a, b).

$f'(x) > 0$ for any $a < x < b$:

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$: **increasing**

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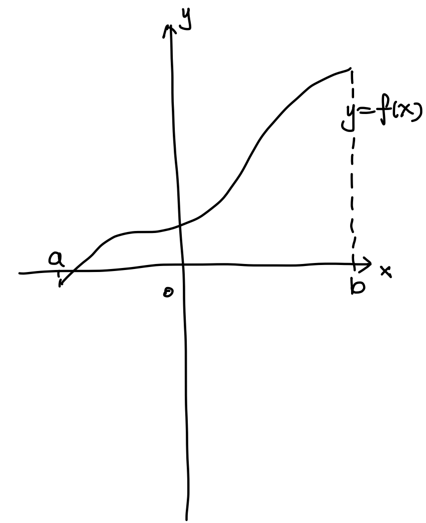

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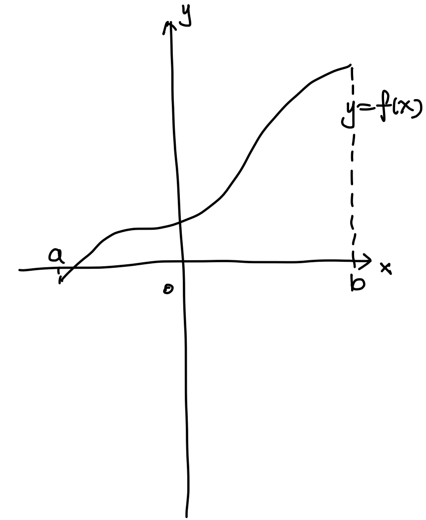

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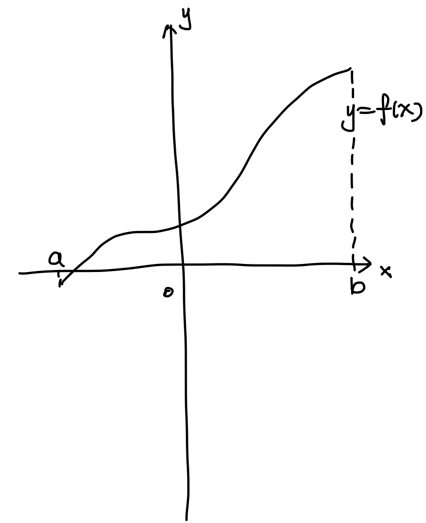

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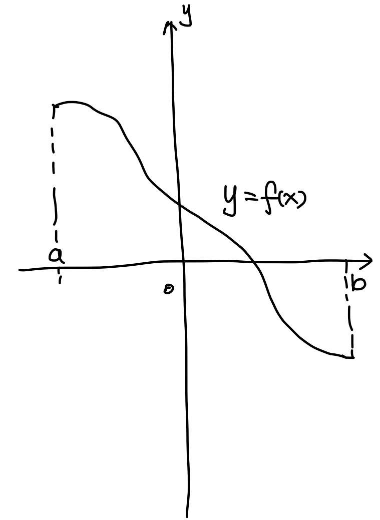

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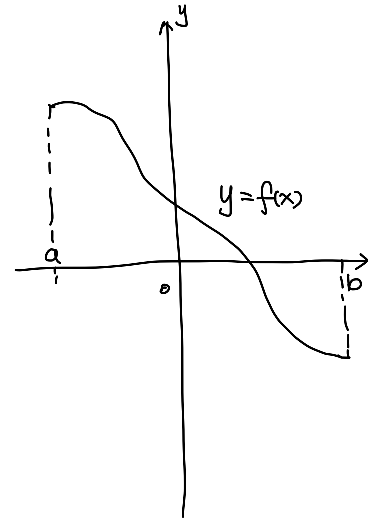

$f''(x)>0$:

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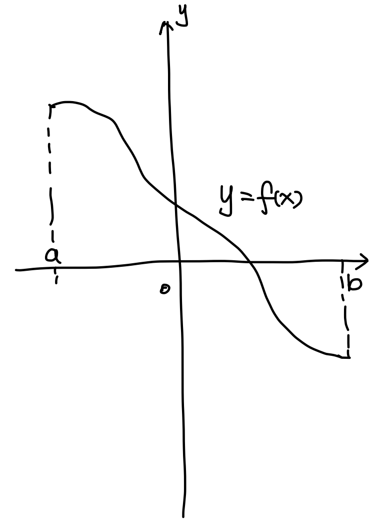

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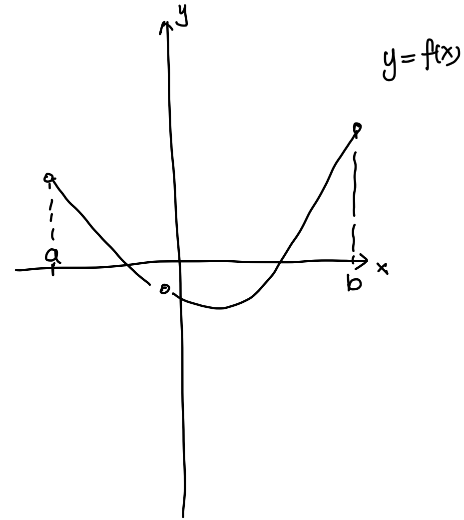

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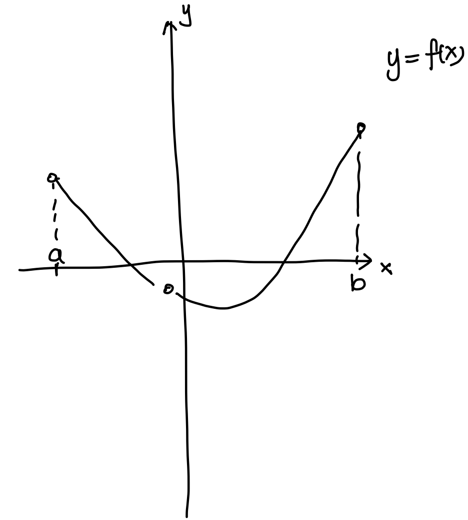

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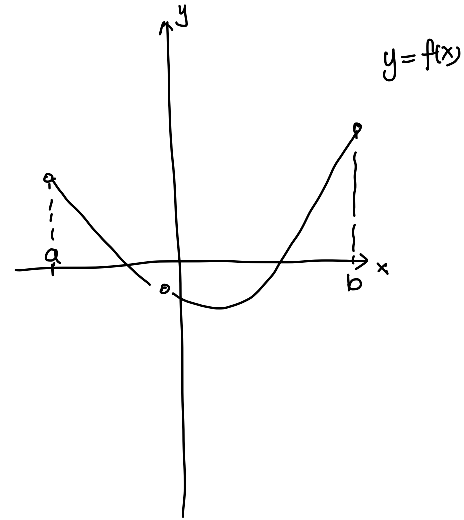

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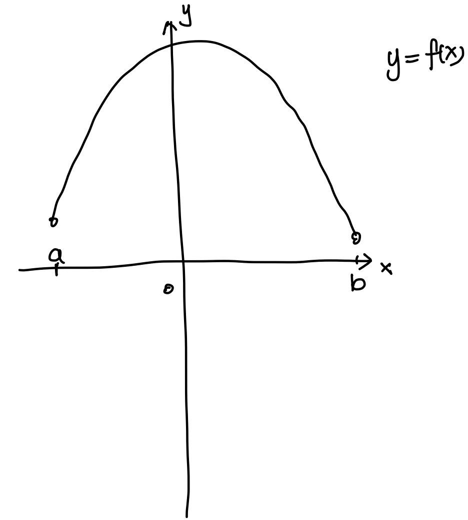

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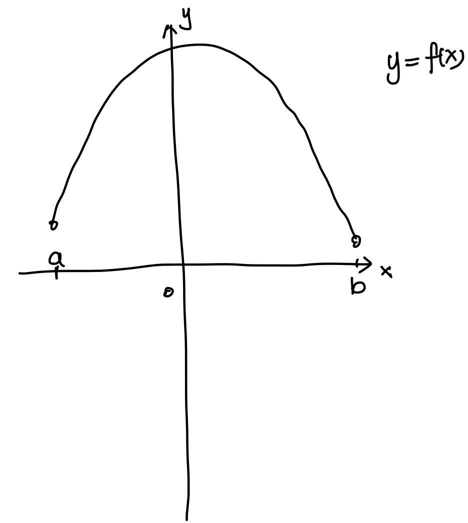

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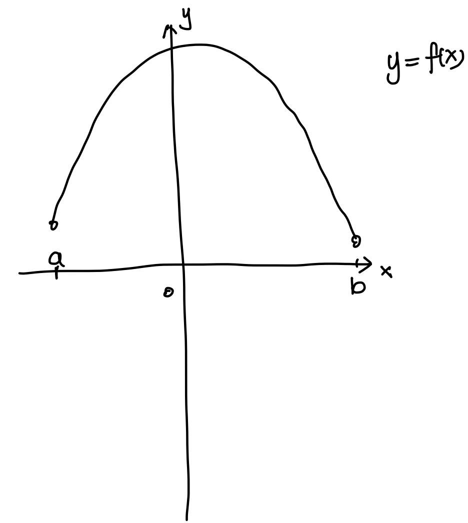

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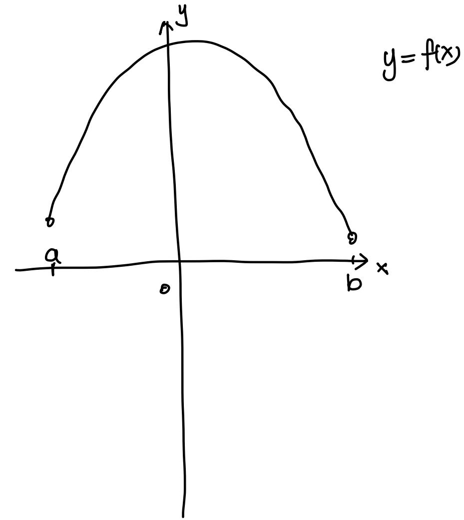

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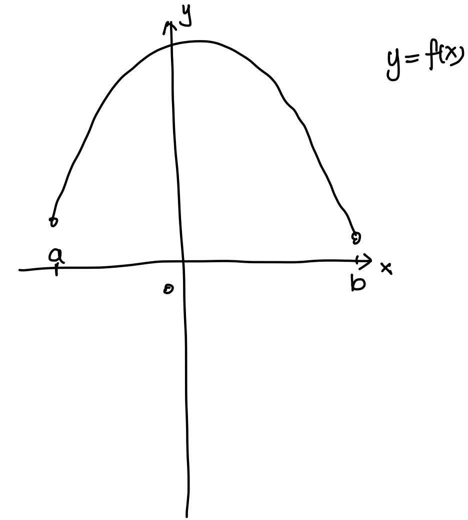

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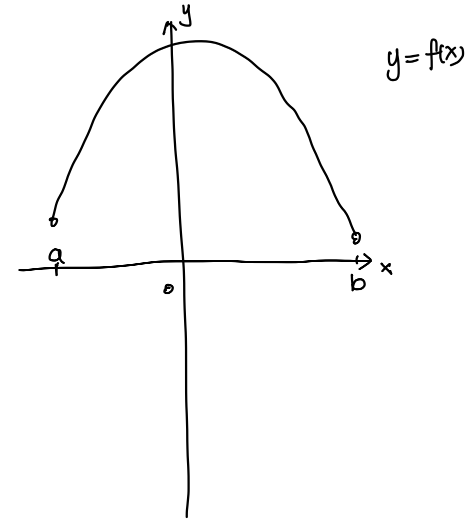

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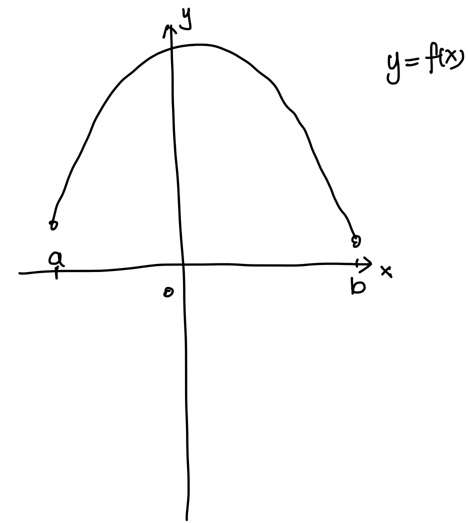

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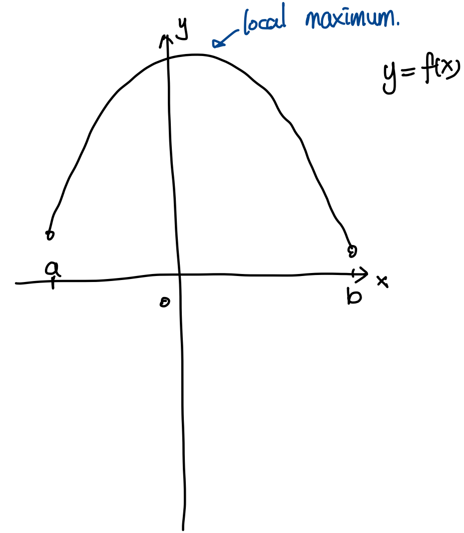

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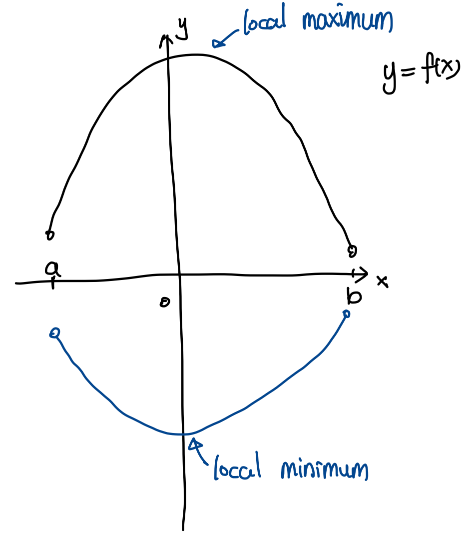

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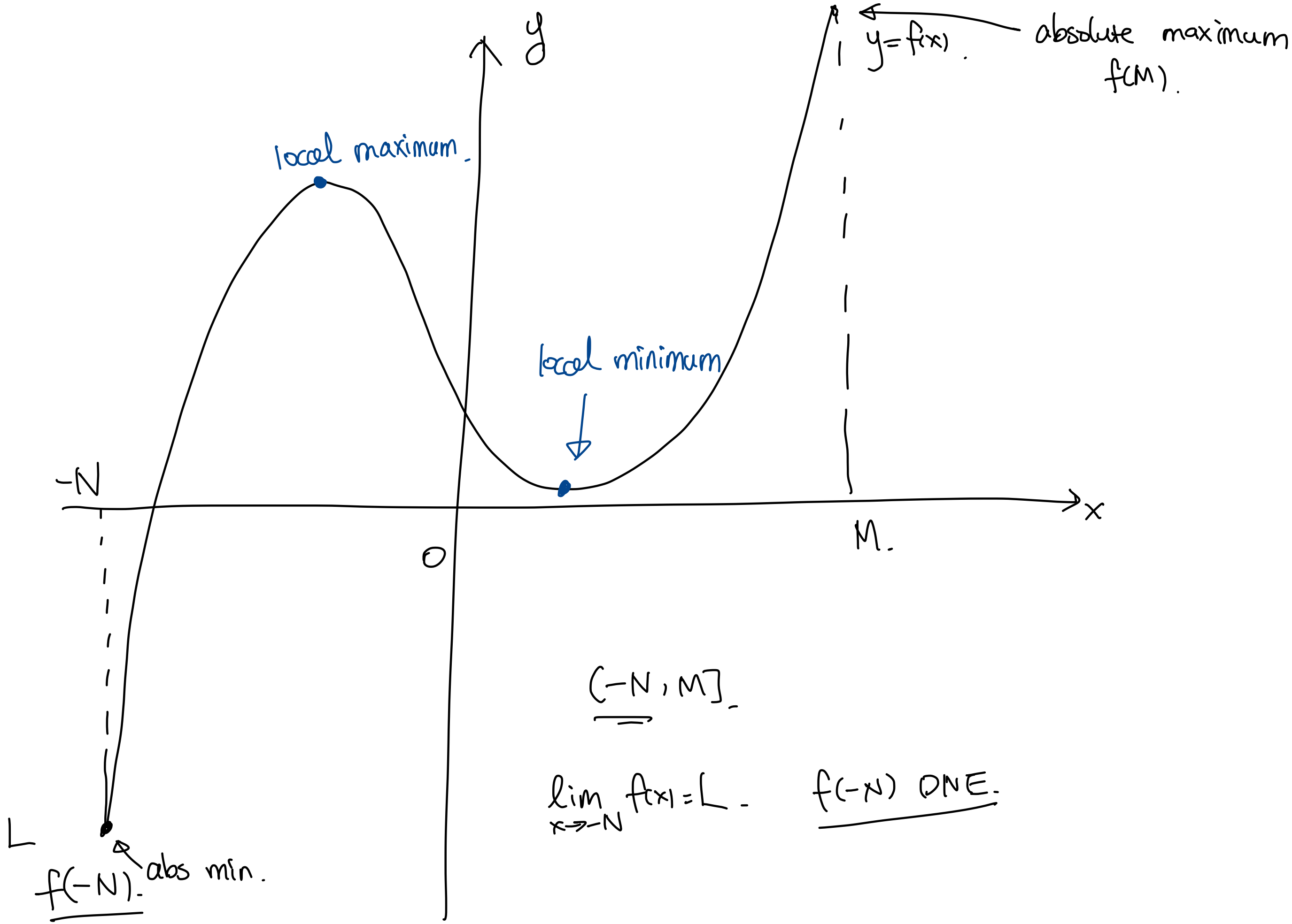
absolute maximum = maximum among all local maxima + value at endpoints.

minimum = minimum minima +

$f'(x) = 0$: **critical**

candidate for extrema

- $f''(x) > 0$: **local minimum**
- $f''(x) < 0$: **local maximum**



$(-N, M]$.

$\lim_{x \to -N} f(x) = L$.

f(−N) DNE.

f : defined in (a,b). cont + differentiable

$\lim_{x\to a} f(x)$ & $\lim_{x\to b} f(x)$ exist.

define. $g(x) = \begin{cases} f(x) \text{ in } (a,b) \\ \lim_{x\to a+} f(x) \text{ at } a. \\ \lim_{x\to b-} f(x) \text{ at } b \end{cases}$ then g is continuous on $[a,b]$ and differentiable in (a,b).

so by MVT, there is $\underline{a<c<b}$ s.t. $g'(c) = \dfrac{g(b)-g(a)}{b-a} = \dfrac{\lim_{x\to b-} f(x) - \lim_{x\to a+} f(x)}{b-a}$.

$$\begin{array}{c} g'(c) \\ \parallel \\ f'(c). \end{array}$$

$$g'(c) = \lim_{h\to 0} \frac{g(c+h)-g(c)}{h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c).$$

pick h small so that $a<c-h<c+h<b$, then

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- Compute the coordinate of local maximum & local minimum.

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- Figure out when $f''(x)>0$, $f''(x)<0$ or $f''(x)=0$.
- Compute the coordinate of local maximum & local minimum.
- Draw the graph following your computation.

Example 1. Draw the graph of the function $y = \frac{x}{x^2+1}$.

$$y' = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} = \frac{-(x^2+1)+2}{(x^2+1)^2} = -\frac{1}{x^2+1} + \frac{2}{(x^2+1)^2}$$

$$y'' = \frac{2x}{(x^2+1)^2} - \frac{2\cdot 2\cdot 2x}{(x^2+1)^3} = \frac{2x(x^2+1)-8x}{(x^2+1)^3} = \frac{2x^3-6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{\underbrace{(x^2+1)^3}_{>0}}$$

$$y' = \frac{-x^2+1}{\underbrace{(x^2+1)^2}_{>0}}$$

$y' > 0 \iff -x^2+1>0$

$x^2<1 \rightsquigarrow -1<x<1$

$y' < 0 \iff -x^2+1<0$

$x^2>1 \rightsquigarrow \begin{matrix} x>1 \\ x<-1 \end{matrix}$

$y' = 0 \iff x_1 = -1, \quad x_2 = 1$ $\qquad (-\infty,-1) \cup (-1,1) \cup (1,+\infty)$

$2x(x^2-3) \begin{matrix} >0 \\ =0 \\ <0 \end{matrix}$ $\quad \sqrt{3}, -\sqrt{3}, 0$

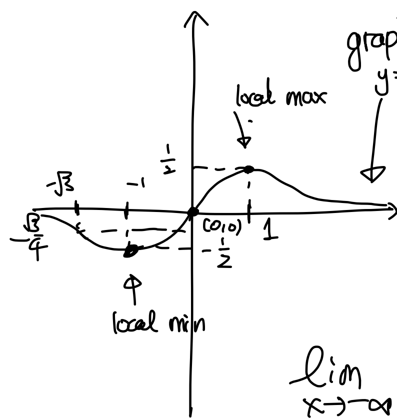
$y' > 0 \iff (-1,1)$

$y' < 0 \iff (-\infty,-1) \cup (1,+\infty)$

$y'' > 0 \iff (-\sqrt{3},0) \cup (\sqrt{3},+\infty)$

$y'' < 0 \iff (-\infty,-\sqrt{3}) \cup (0,\sqrt{3})$

$y'' = 0 \iff \sqrt{3}, -\sqrt{3}, 0$



graph of $y = \frac{x}{x^2+1}$.

$y''<0$	+	−	+
$(-\infty, -\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},+\infty)$
−			

$-\sqrt{3} \qquad 0 \qquad \sqrt{3}$.

$x < -\sqrt{3}$: $\quad 2x<0 \quad x^2-3>0$.

sign: −

$$\lim_{x\to-\infty} \frac{x}{x^2+1} = 0.$$

$\frac{x}{x^2+1} < 0$

as $x<0$.

Example 2. Draw the graph of the function $f(x) = 2\sin x + \sin^2 x$ on $[0, 2\pi]$.

Mean Value Theorems

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Rolle's Theorem. $f(x)$ continuous on $[a,b]$ and differentiable in (a,b), $f(a)=f(b)$, then there exists $c\in(a,b)$, $f'(c)=0$;

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Mean Value Theorem. $f(x)$ continuous on $[a,b]$ and differentiable in (a,b), then there is $a<c<b$ so that $\dfrac{f(b)-f(a)}{b-a}=f'(c)$.

Example 3: Show that the equation $x^4+4x^3+c=0$ has at most 2 real roots.

Linear Approximation.

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- Big O and small o notation

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Intuitively: $f(x) = O(g(x)) \longleftrightarrow$ f grows NOT faster than g / g descends NOT faster than f.

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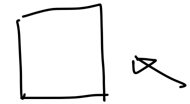
Intuitively: $f(x) = O(g(x)) \longleftrightarrow$ f grows NOT faster than g / g descends NOT faster than f.

$f(x) = o(g(x)) \longleftrightarrow$ f grows slower than g / f descends faster than g.

Example 4. Compare $f(x) = \frac{1}{x}$ and $g(x) = \frac{\sin x}{x^2+1}$ at $x=0$ and $x = +\infty$.

Announcements :



squares.

- Quiz next Thursday (3 problems, one for MVT, one for drawing graphs, one for related rates)

  Practice problem : homework & midterm

- Office hour : today, 1-3pm (additional appointment accepted via email)

See U Next Week!