Oct. 21st

MATH 125.

Outline

Outline

- Drawing graphs of functions
- Mean Value Theorem Again
- (Optional) Linear Approximation

Drawing Graphs of Functions.

Drawing Graphs of Functions.

Review: In chapter one, we learn how to draw the graph of some function:

Drawing Graphs of Functions.

Review: In chapter one, we learn how to draw the graph of some function:

△ Well-known models: $y = \frac{1}{x}$, $y = \sin x$, $y = x^2 + ax + b$, $a, b \in \mathbb{R}$, ...

Drawing Graphs of Functions.

Review: In chapter one, we learn how to draw the graph of some function:

△ Well-known models: $y=\frac{1}{x}$, $y=\sin x$, $y=x^2+ax+b$, $a,b\in\mathbb{R}$, $\cdots$

△ Elementary transformations: $f(x+c)$, $f(cx)$, $f(x)+c$, $cf(x)$,

Drawing Graphs of Functions.

Review: In chapter one, we learn how to draw the graph of some function:

△ Well-known models : $y=\frac{1}{x}$, $y=\sin x$, $y=x^2+ax+b$, $a,b\in\mathbb{R}$, ...

△ Elementary transformations: $f(x+c)$, $f(cx)$, $f(x)+c$, $cf(x)$,

$\updownarrow$

Drawing Graphs of Functions.

Review: In chapter one, we learn how to draw the graph of some function:

△ Well-known models : $y = \frac{1}{x}$, $y = \sin x$, $y = x^2 + ax + b$, $a, b \in \mathbb{R}$, ...

△ Elementary transformations: $f(x+c)$, $f(cx)$, $f(x)+c$, $cf(x)$,

$\updownarrow$

operations on graphs of functions

Drawing Graphs of Functions.

Review: In chapter one, we learn how to draw the graph of some function:

△ Well-known models: $y=\frac{1}{x}$, $y=\sin x$, $y=x^2+ax+b$, $a,b\in\mathbb{R}$, $\cdots$

△ Elementary transformations: $f(x+c)$, $f(cx)$, $f(x)+c$, $cf(x)$,

$\updownarrow$

operations on graphs of functions

Now: derivative $\rightsquigarrow$ monotonicity & convexity.

Monotonicity and Convexity

Monotonicity and Convexity

$y=f(x)$ differentiable on (a,b).

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$:

Monotonicity and Convexity

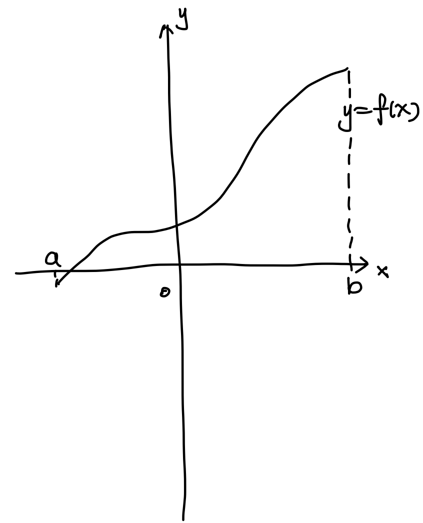
$y = f(x)$ differentiable in (a, b).

$f'(x) > 0$ for any $a < x < b$: increasing

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$: **increasing**

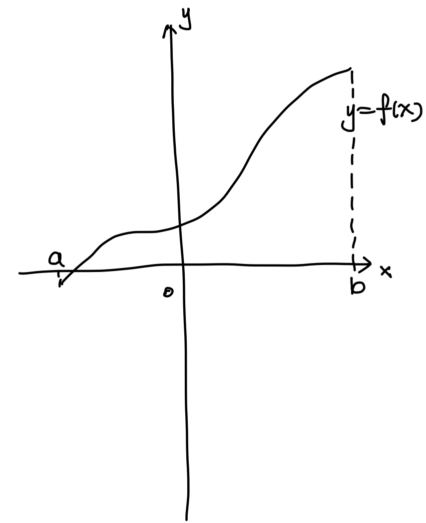


Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$: increasing

$f'(x)<0$ for any $a<x<b$:

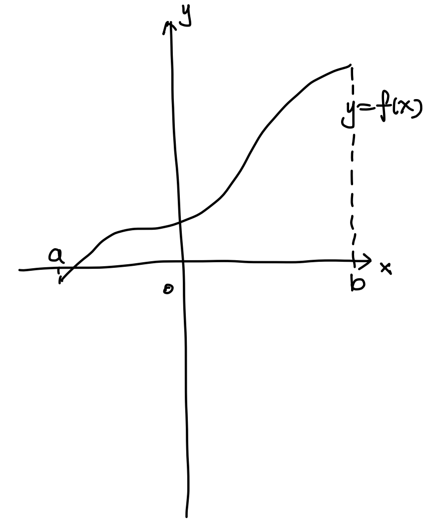


Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$: **increasing**

$f'(x)<0$ for any $a<x<b$: **decreasing**

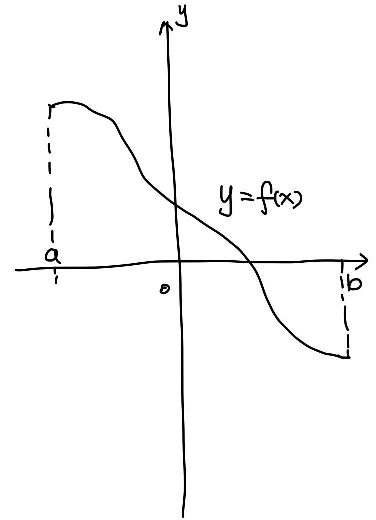


Monotonicity and Convexity

$y = f(x)$ differentiable in (a, b).

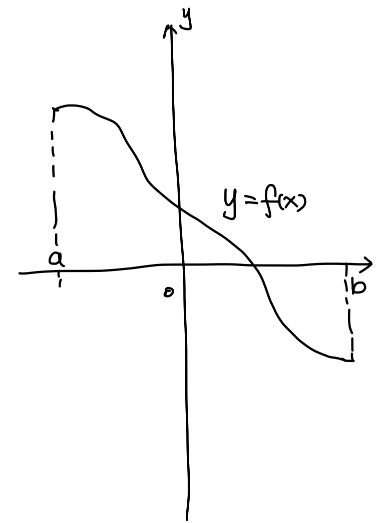
$f'(x) > 0$ for any $a < x < b$: increasing

$f'(x) < 0$ for any $a < x < b$: decreasing



Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).



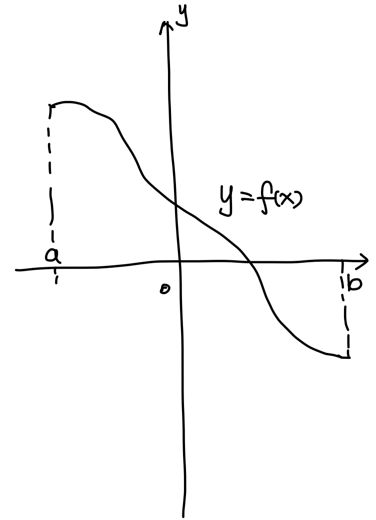
$f'(x)>0$ for any $a<x<b$: **increasing**

$f'(x)<0$ for any $a<x<b$: **decreasing**

$f''(x)>0$:

Monotonicity and Convexity

$y=f(x)$ differentiable in (a, b).



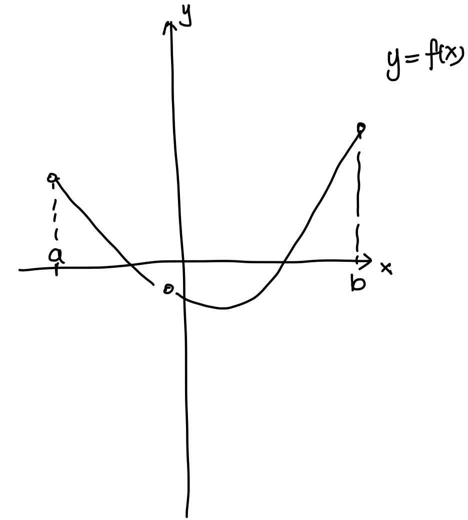
$f'(x)>0$ for any $a<x<b$: increasing

$f'(x)<0$ for any $a<x<b$: decreasing

$f''(x)>0$: convex

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).



$f'(x)>0$ for any $a<x<b$: **increasing**

$f'(x)<0$ for any $a<x<b$: **decreasing**

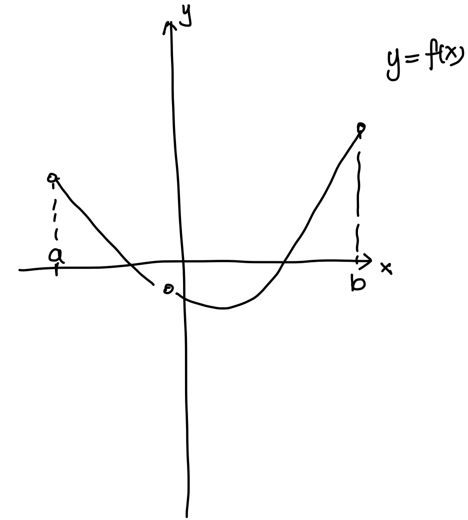
$f''(x)>0$: **convex**

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$: **increasing**

$f'(x)<0$ for any $a<x<b$: **decreasing**



$f''(x)>0$: **convex**

$f''(x)<0$:

Monotonicity and Convexity

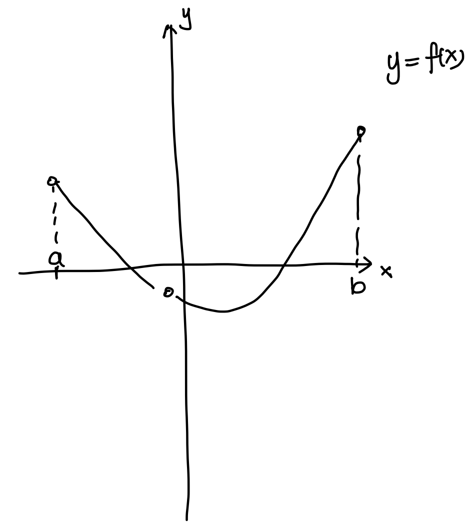
$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$: increasing

$f'(x)<0$ for any $a<x<b$: decreasing

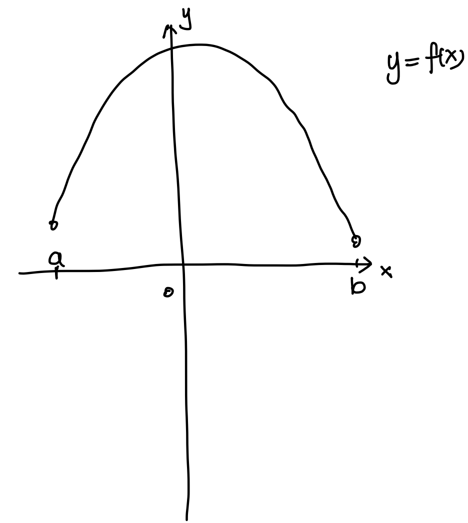
$f''(x)>0$: convex

$f''(x)<0$: concave



Monotonicity and Convexity

$y=f(x)$ differentiable in (a, b).



$f'(x)>0$ for any $a<x<b$: **increasing**

$f'(x)<0$ for any $a<x<b$: **decreasing**

$f''(x)>0$: **convex**

$f''(x)<0$: **concave**

Monotonicity and Convexity

$y = f(x)$ differentiable in (a, b).

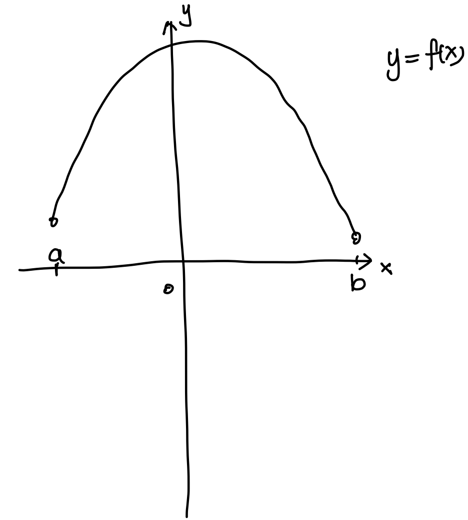
$f'(x) > 0$ for any $a < x < b$: increasing

$f'(x) < 0$ for any $a < x < b$: decreasing

$f''(x) > 0$: convex

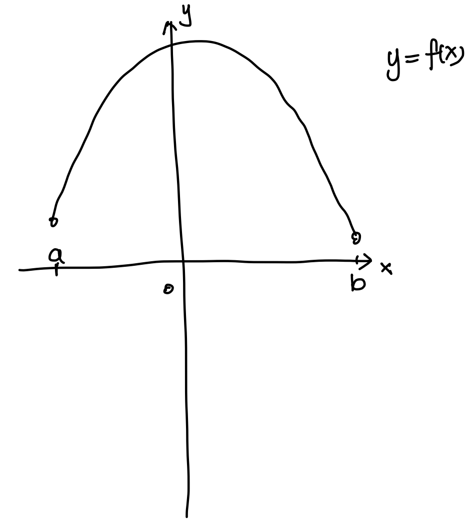
$f''(x) < 0$: concave

$f'(x) = 0$:



Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).



$f'(x)>0$ for any $a<x<b$: **increasing**

$f'(x)<0$ for any $a<x<b$: **decreasing**

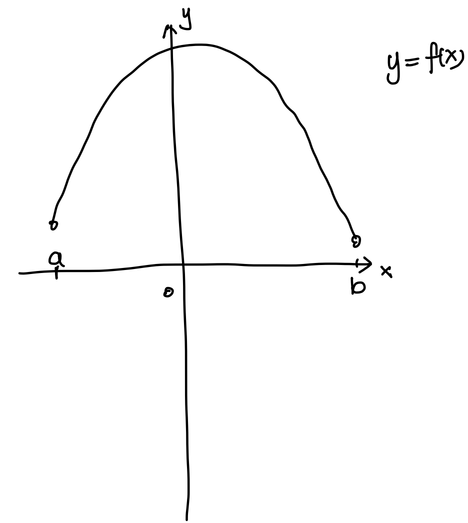
$f''(x)>0$: **convex**

$f''(x)<0$: **concave**

$f'(x)=0$: **critical**

Monotonicity and Convexity

$y = f(x)$ differentiable in (a, b).



$f'(x) > 0$ for any $a < x < b$: **increasing**

$f'(x) < 0$ for any $a < x < b$: **decreasing**

$f''(x) > 0$: **convex**

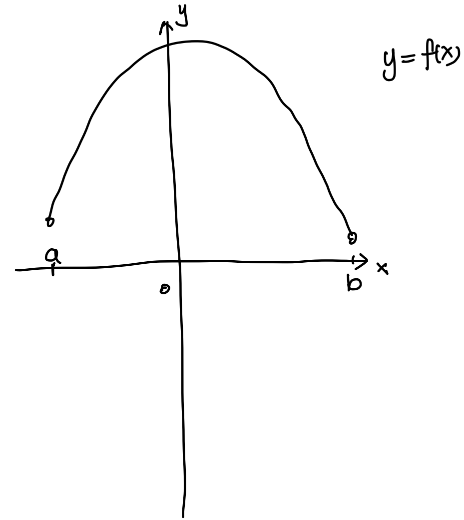
$f''(x) < 0$: **concave**

$f'(x) = 0$: **critical**

candidate for extrema

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).



$f'(x)>0$ for any $a<x<b$: **increasing**

$f'(x)<0$ for any $a<x<b$: **decreasing**

$f''(x)>0$: **convex**

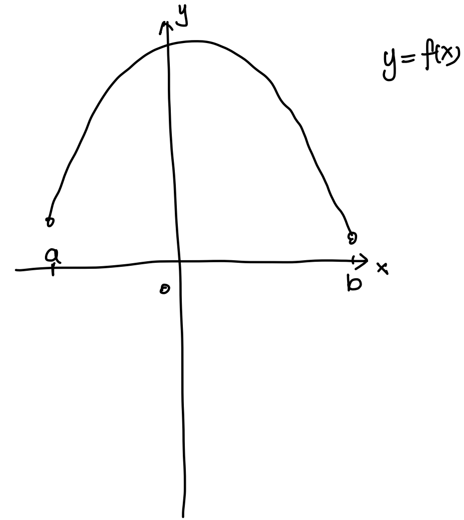
$f''(x)<0$: **concave**

$f'(x)=0$: **critical**

candidate for extrema $\begin{cases} f''(x)>0 : \\ f''(x)<0 . \end{cases}$

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).



$f'(x)>0$ for any $a<x<b$: **increasing**

$f'(x)<0$ for any $a<x<b$: **decreasing**

$f''(x)>0$: **convex**

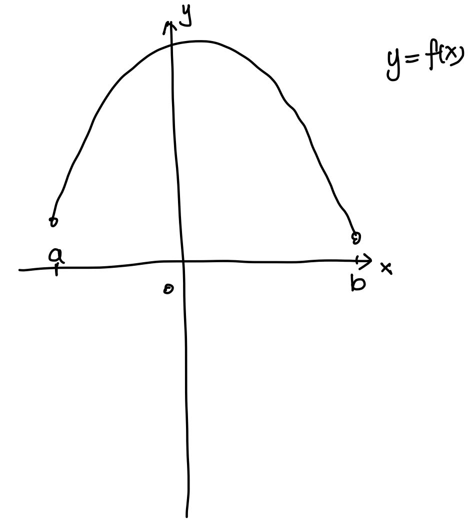
$f''(x)<0$: **concave**

$f'(x)=0$: **critical**

candidate for extrema $\begin{cases} f''(x)>0 : \text{local minimum} \\ f''(x)<0 : \end{cases}$

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).



$f'(x)>0$ for any $a<x<b$: increasing

$f'(x)<0$ for any $a<x<b$: decreasing

$f''(x)>0$: convex

$f''(x)<0$: concave

$f'(x)=0$: critical

candidate for extrema $\begin{cases} f''(x)>0 : \text{local minimum} \\ f''(x)<0 : \text{local maximum} \end{cases}$

Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$: **increasing**

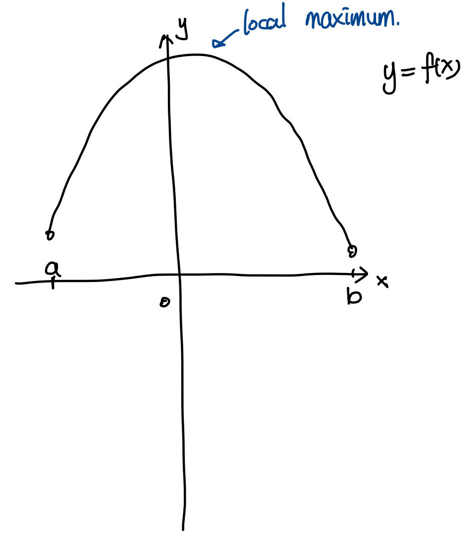
$f'(x)<0$ for any $a<x<b$: **decreasing**

$f''(x)>0$: **convex**

$f''(x)<0$: **concave**

$f'(x)=0$: **critical**

candidate for extrema $\begin{cases} f''(x)>0 : \textbf{local minimum} \\ f''(x)<0 : \textbf{local maximum} \end{cases}$



Monotonicity and Convexity

$y=f(x)$ differentiable in (a,b).

$f'(x)>0$ for any $a<x<b$: **increasing**

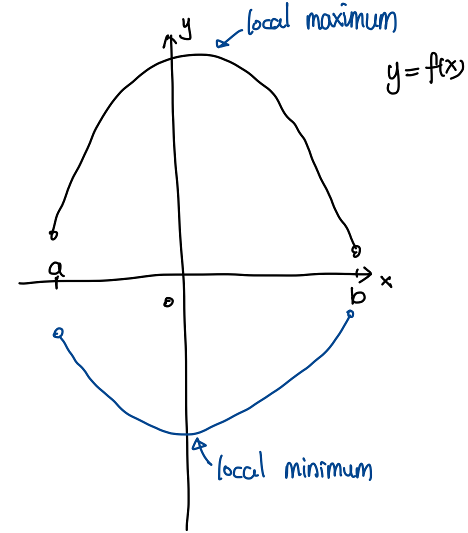
$f'(x)<0$ for any $a<x<b$: **decreasing**

$f''(x)>0$: **convex**

$f''(x)<0$: **concave**

$f'(x)=0$: **critical**

candidate for extrema
$$\begin{cases} f''(x)>0 : \text{local minimum} \\ f''(x)<0 : \text{local maximum} \end{cases}$$



How to Draw a Graph?

How to Draw a Graph?

- Compute $f'(x)$, $f''(x)$

How to Draw a Graph?

- Compute $f'(x)$, $f''(x)$
- Figure out the interval when $f'(x)>0$, $f'(x)<0$, or $f'(x)=0$.

How to Draw a Graph?

- Compute $f'(x)$, $f''(x)$
- Figure out the interval when $f'(x)>0$, $f'(x)<0$, or $f'(x)=0$.
- Figure out when $f''(x)>0$, $f''(x)<0$ or $f''(x)=0$. ← point of inflection.

How to Draw a Graph?

- Compute $f'(x)$, $f''(x)$
- Figure out the interval when $f'(x)>0$, $f'(x)<0$, or $f'(x)=0$.
- Figure out when $f''(x)>0$, $f''(x)<0$ or $f''(x)=0$.
- Compute the coordinate of local maximum & local minimum.

  all the critical & inflection points.

How to Draw a Graph?

- Compute $f'(x)$, $f''(x)$
- Figure out the interval when $f'(x)>0$, $f'(x)<0$, or $f'(x)=0$.
- Figure out when $f''(x)>0$, $f''(x)<0$ or $f''(x)=0$.
- Compute the coordinate of local maximum & local minimum.
- Draw the graph following your computation.

Example 1. Draw the graph of the function $y = \frac{x}{x^2+1} =: f(x)$

$$y' = \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

$$y'' = \left(\frac{-x^2+1}{(x^2+1)^2}\right)' = \left(\frac{-(x^2+1)+2}{(x^2+1)^2}\right)' = \left(-\frac{1}{x^2+1} + \frac{2}{(x^2+1)^2}\right)'$$

$$= \frac{2x}{(x^2+1)^2} - \frac{2\cdot 2\cdot 2x}{(x^2+1)^3} = \frac{2x}{(x^2+1)^2} - \frac{8x}{(x^2+1)^3}$$

$$= \frac{2x(x^2+1) - 8x}{(x^2+1)^3} = \frac{2x^3+2x-8x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = 0$$

$$\lim_{x\to-\infty} \frac{x}{x^2+1} = 0$$

$x<0$, then $\frac{x}{x^2+1}<0$.

$x>0$, then $\frac{x}{x^2+1}>0$.

$y''<0$: $(-\infty,-\sqrt{3}) \cup (0,\sqrt{3})$.

$y'=0$: $1, -1$; $\longrightarrow$ 1 is a local maximum, $f(1) = \frac{1}{2}$

-1 is a local minimum. $f(-1) = -\frac{1}{2}$

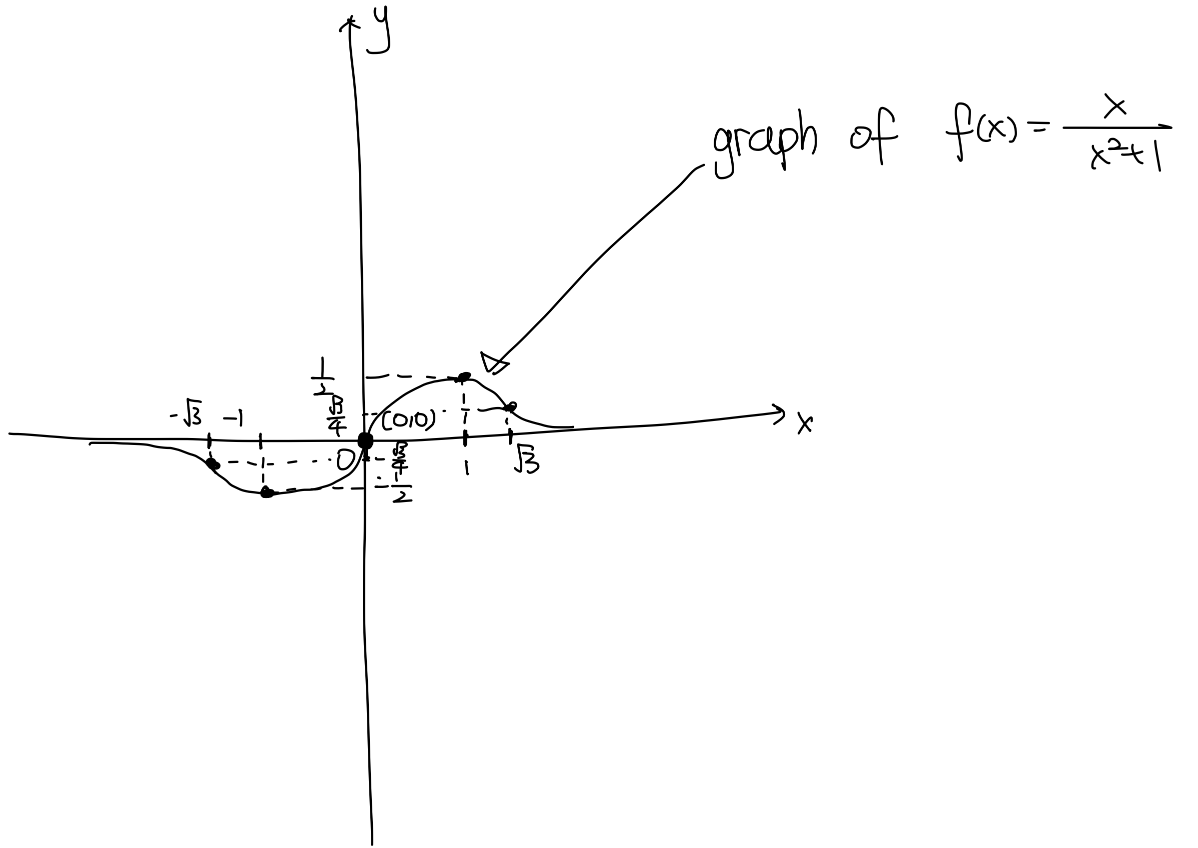
$y'>0$: $\cancel{x=0}$ $(-1,1)$

$y'<0$: $\cancel{\text{DNE}}$ $\{x>1\} \cup (-\infty,-1)$

$y''=0$: $0, \sqrt{3}, -\sqrt{3}$; $\longrightarrow$ $f(0)=0$, $f(\sqrt{3}) = \frac{\sqrt{3}}{4}$, $f(-\sqrt{3}) = -\frac{\sqrt{3}}{4}$.

$y''>0$: $(-\sqrt{3},0) \cup (\sqrt{3},+\infty)$

Example 1. Draw the graph of the function $y = \frac{x}{x^2+1}$.



Example 1. Draw the graph of the function $y=\frac{x}{x^2+1}$.

$$y' = \frac{-x^2+1}{(x^2+1)^2}$$

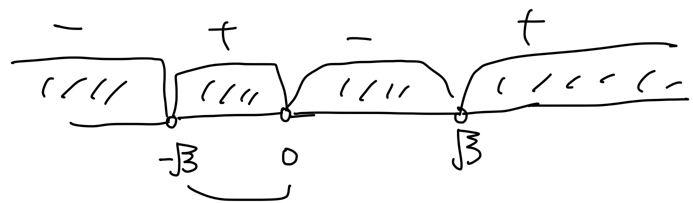
$y'>0 \rightsquigarrow -x^2+1>0 \rightsquigarrow x^2<1 \rightsquigarrow (-1,1)$

$y'<0 \rightsquigarrow -x^2+1<0 \rightsquigarrow x^2>1 \rightsquigarrow (1,+\infty)\cup(-\infty,-1)$

$y''>0 \rightsquigarrow 2x(x^2-3)>0 \qquad (-\sqrt{3},0)\cup(\sqrt{3},+\infty)$

$y''<0 \rightsquigarrow \qquad (-\infty,-\sqrt{3})\cup(0,\sqrt{3})$.

Trick: $\mathbb{R}$. remove all zeros of f'':



when $x<-\sqrt{3}$. $2x<0$, $x^2-3>0 \rightsquigarrow y''<0$

when $-\sqrt{3}<x<0$: $2x<0$ $x^2-3<0$. $\rightsquigarrow y''>0$.

when $0<x<\sqrt{3}$: $2x>0$, $x^2-3<0 \rightsquigarrow y''<0$

when $x>\sqrt{3}$: $2x>0$, $x^2-3>0 \rightsquigarrow y''>0$

Example 2. Draw the graph of the function $f(x) = 2\sin x + \sin^2 x$ on $[0, 2\pi]$.

Mean Value Theorems

Mean Value Theorems

Rolle's Theorem. $f(x)$ continuous on $[a,b]$ and differentiable in (a,b), $f(a)=f(b)$, then there exists $c\in(a,b)$, $f'(c)=0$;

Mean Value Theorems

Rolle's Theorem. $f(x)$ continuous on $[a,b]$ and differentiable in (a,b), $f(a)=f(b)$, then there exists $a<c<b$, $f'(c)=0$;

Mean Value Theorem. $f(x)$ continuous on $[a,b]$ and differentiable in (a,b), then there is $a<c<b$ so that $\frac{f(b)-f(a)}{b-a}=f'(c)$.

Example 3: Show that the equation $x^4+4x^3+c=0$ has at most 2 real roots.

Linear Approximation.

Linear Approximation.

- Big O and small o notation

Linear Approximation.

- Big O and small o notation

$f(x) = O(g(x))$ as $x \to a$ (or ∞) means

Linear Approximation.

- Big O and small o notation

  $f(x) = O(g(x))$ as $x \to a$ (or ∞) means there is $C > 0$ so that

  $|f(x)| \leq C|g(x)|$ if x is sufficiently closed to a.

Linear Approximation.

- Big O and small o notation

$f(x) = O(g(x))$ as $x \to a$ (or ∞) means there is $C > 0$ so that

$|f(x)| \leq C|g(x)|$ if x is sufficiently closed to a.

$f(x) = o(g(x))$ as $x \to a$ means

Linear Approximation.

- Big O and small o notation

$f(x) = O(g(x))$ as $x \to a$ (or ∞) means there is $C > 0$ so that

$|f(x)| \le C|g(x)|$ if x is sufficiently closed to a.

$f(x) = o(g(x))$ as $x \to a$ (or ∞) means $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

Linear Approximation.

- Big O and small o notation

$f(x) = O(g(x))$ as $x \to a$ (or ∞) means there is $C > 0$ so that

$|f(x)| \le C|g(x)|$ if x is sufficiently closed to a.

$f(x) = o(g(x))$ as $x \to a$ (or ∞) means $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

Intuitively :

Linear Approximation.

- Big O and small o notation

$f(x) = O(g(x))$ as $x \to a$ (or ∞) means there is $C > 0$ so that

$|f(x)| \leq C|g(x)|$ if x is sufficiently closed to a.

$f(x) = o(g(x))$ as $x \to a$ (or ∞) means $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

Intuitively: $f(x) = O(g(x)) \longleftrightarrow$

Linear Approximation.

- Big O and small o notation

$f(x) = O(g(x))$ as $x \to a$ (or ∞) means there is $C>0$ so that

$|f(x)| \le C|g(x)|$ if x is sufficiently closed to a.

$f(x) = o(g(x))$ as $x \to a$ (or ∞) means $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$

Intuitively: $f(x) = O(g(x)) \longleftrightarrow$ f grows NOT faster than g / g descends NOT faster than f.

$\lim_{x\to a} f(x) = \infty \rightsquigarrow \lim_{x\to a} g = \infty$

$\lim_{x\to a} g(x) = 0 \rightsquigarrow \lim_{x\to a} f(x) = 0$.

Linear Approximation.

- Big O and small o notation

$f(x) = O(g(x))$ as $x \to a$ (or ∞) means there is $C > 0$ so that

$|f(x)| \le C|g(x)|$ if x is sufficiently closed to a.

$f(x) = o(g(x))$ as $x \to a$ (or ∞) means $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

Intuitively: $f(x) = O(g(x)) \longleftrightarrow$ f grows NOT faster than g / g descends NOT faster than f.

$f(x) = o(g(x)) \longleftrightarrow$

Linear Approximation.

- Big O and small o notation

$f(x) = O(g(x))$ as $x \to a$ (or ∞) means there is $C > 0$ so that

$|f(x)| \leq C|g(x)|$ if x is sufficiently closed to a.

$f(x) = o(g(x))$ as $x \to a$ (or ∞) means $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$

Intuitively: $f(x) = O(g(x)) \longleftrightarrow$ f grows NOT faster than g / g descends NOT faster than f.

$f(x) = o(g(x)) \longleftrightarrow$ f grows slower than g / f descends faster than g.

Example 4. Compare $f(x) = \frac{1}{x}$ and $g(x) = \frac{\sin x}{x^2+1}$ at $x=0$ and $x=+\infty$.

$$\underline{\lim_{x\to 0} \frac{f}{g}} = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{\sin x}{x^2+1}} = \lim_{x\to 0} \frac{1}{\frac{x\sin x}{x^2+1}} = \underline{\lim_{x\to 0} \frac{x^2+1}{x\sin x}} = \infty$$

$$\lim_{x\to 0} \frac{g}{f} = 0 \quad \rightsquigarrow \quad g(x) = o(f(x)) \text{ as } x\to 0$$

$$\lim_{x\to +\infty} \frac{g}{f} = \lim_{x\to +\infty} \frac{x\sin x}{x^2+1} = 0 \quad \rightsquigarrow \quad g(x) = o(f(x)) \text{ as } x\to +\infty.$$

Announcements :

← squares.

- Quiz next Thursday (3 problems, one for MVT, one for drawing graphs, one for related rates)

  Practice problem : homework & midterm

- Office hour : today, 1-3pm (additional appointment accepted via email)

See U Next Week!