Nov. 9th, 2021

Integrations

Siyang Liu

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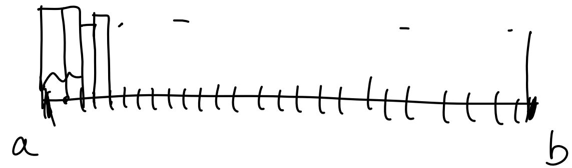

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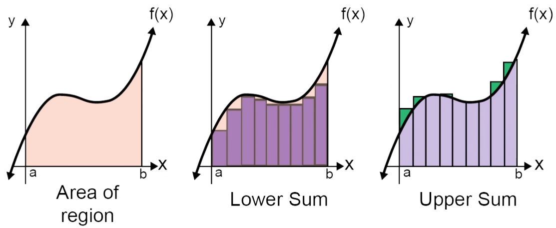

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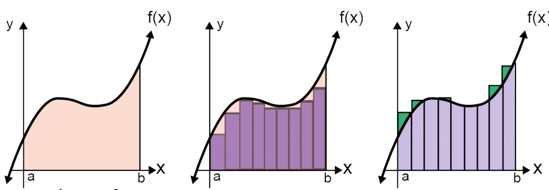
lower sum : pick x_i^* so that $f(x_i^*)$ is smallest.

upper sum : pick x_i^* so that $f(x_i^*)$ largest.



Area of region

Lower Sum

Upper Sum

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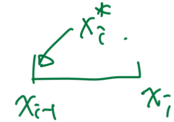
Problem 6 from Spring 2020 final exam: Consider the integral $\int_0^1 \sqrt{1+x^2}\,dx$.

$f(x) = \sqrt{1+x^2}$.

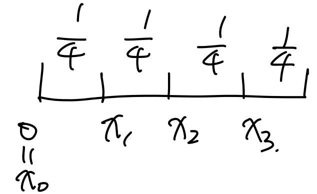
we pick x_i^* to be x_{i-1}.



a. Express the integral as a left Riemann sum with 4 sub-intervals of equal width. You may leave your answer as an unspecified sum

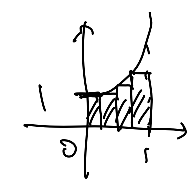
b. Is the above Riemann sum in a. greater than or less than the value of the integral? Briefly explain.

a.) $\int_0^1 \sqrt{1+x^2}\,dx = \lim\limits_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ $\quad n=4,\quad x_0=0,\ x_1=\frac{1}{4},\ x_2=\frac{1}{2},\ x_3=\frac{3}{4},\ x_4=1.$



$$S_4 = f(x_0)\underbrace{\Delta x_1}_{\frac{1}{4}} + f(x_1)\Delta x_2 + f(x_2)\Delta x_3 + f(x_3)\Delta x_4$$

$$= \sqrt{1+0^2}\cdot\frac{1}{4} + \sqrt{1+\left(\frac{1}{4}\right)^2}\cdot\frac{1}{4} + \sqrt{1+\left(\frac{1}{2}\right)^2}\,\frac{1}{4} + \sqrt{1+\left(\frac{3}{4}\right)^2}\cdot\frac{1}{4}.$$



b.) $f(x)=\sqrt{1+x^2}$ is increasing. x_{i-1} is the minimum of f inside $[x_{i-1}, x_i]$. $\rightsquigarrow$ lower sum. this sum is less than the value of the integral.

$\int_0^1 \sqrt{1+x^2}\, dx.$ write $x = \tan u.$ $d(\tan u) = \sec^2 u\, du.$

$$\int_0^1 \sqrt{1+\tan^2 u}\; d(\tan u) = \int_0^{\frac{\pi}{4}} \underline{\sqrt{\sec^2 u}}.\ \sec^2 u\, du.$$

$0 \le \tan u \le 1$

$0 \le u \le \frac{\pi}{4}.$

$$= \int_0^{\frac{\pi}{4}} \sec u \cdot \sec^2 u\, du = \int_0^{\frac{\pi}{4}} \sec^3 u\, du$$

$$= \int_0^{\frac{\pi}{4}} \frac{du}{\cos^3 u}.$$

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- fundamental theorem of Calculus : $\int_a^b f'(x)\,dx = f(b) - f(a)$;
- substitution law : $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

Problem 2.b from Spring 2020 Final: evaluate the integral

$$I=\int_{-\frac{1}{k}}^{\frac{3}{k}} x\sqrt{kx+1}\,dx\,,\quad \text{where } k \text{ is a nonzero constant.}$$

$\sqrt{(kx+1)^3}\,.$

$$I=\int_{-\frac{1}{k}}^{\frac{3}{k}} \frac{1}{k}\cdot kx\sqrt{kx+1}\,dx=\frac{1}{k}\int_{-\frac{1}{k}}^{\frac{3}{k}} kx\sqrt{kx+1}\,dx\,.=\frac{1}{k}\int_{-\frac{1}{k}}^{\frac{3}{k}}\left((kx+1)\sqrt{kx+1}-\sqrt{kx+1}\right)dx$$

$$=\frac{1}{k}\int_{-\frac{1}{k}}^{\frac{3}{k}}\sqrt{(kx+1)^3}\,dx-\frac{1}{k}\int_{-\frac{1}{k}}^{\frac{3}{k}}\sqrt{kx+1}\,dx\,.$$

$u=kx+1$, then $\quad -\frac{1}{k}\leq x\leq\frac{3}{k}.$

$du=k\,dx\,.\quad$ then

$dx=\frac{1}{k}du\quad -1\leq kx\leq 3$

$0\leq kx+1\leq 4$

$$=\frac{1}{k}\int_0^4\sqrt{u^3}\,\frac{du}{k}-\frac{1}{k}\int_0^4\sqrt{u}\,\frac{du}{k}.$$

$\sqrt{u^3}=u^{\frac{3}{2}}.$

$\sqrt{u}=u^{\frac{1}{2}}$

$$=\frac{1}{k^2}\cdot\frac{2}{5}\,u^{\frac{5}{2}}\Big|_0^4-\frac{1}{k^2}\cdot\frac{2}{3}\,u^{\frac{3}{2}}\Big|_0^4=\frac{1}{k^2}\cdot\frac{2}{5}\cdot\left(4^{\frac{5}{2}}-0\right)-\frac{1}{k^2}\cdot\frac{2}{3}\cdot\left(4^{\frac{3}{2}}-0\right)=$$

$$= \frac{1}{k^2}\cdot\frac{2}{5}\cdot\sqrt{4^5} - \frac{1}{k^2}\cdot\frac{2}{3}\cdot\sqrt{4^3} = \frac{1}{k^2}\frac{2}{5}\cdot 2^5 - \frac{1}{k^2}\cdot\frac{2}{3}\cdot 2^3.$$

$$= \frac{64}{5k^2} - \frac{16}{3k}.$$

$$= \frac{64\times3-16\times5}{15k^2}.$$

$$= \frac{192-80}{15k2} = \frac{112}{15k^2}.$$

$-\frac{1}{k} \le x \le \frac{3}{k}$

then $0 \le kx+1 \le 4$

then $0 \le u \le 2$

$$\int_{-\frac{1}{k}}^{\frac{3}{k}} x\sqrt{kx+1}\,dx$$

$u=\sqrt{kx+1}$, then $du = \frac{k}{2\sqrt{kx+1}}dx$

$u^2=kx+1$

$x=\frac{u^2-1}{k}$ $\quad dx=\frac{2\sqrt{kx+1}}{k}du$

$$= \frac{2u}{k}du$$

$$= \int_0^2 \frac{u^2-1}{k}\cdot u\cdot\frac{2u}{k}du$$

$$= \frac{1}{k^2}\int_0^2 2u^2(u^2-1)du = \frac{1}{k^2}\int_0^2 (2u^4-2u^2)\,du.$$

Reminder:

- Quiz 7 on Thursday: Riemann sum (8 pts), Problem 6 of Spring 2020 Final

  Substitition law (8 pts), Problem 2 b. of Spring 2020 Fall;

  Evaluating limit (8 pts), Problem 1 b. of Spring 2020 Fall.

- OH: 1-2 pm TODAY. Appointments accepted.

  Grade questions: $\uparrow$ Michael.

See You on Thursday!