Nov. 9^{th}, 2021

Integrations

Siyang Liu

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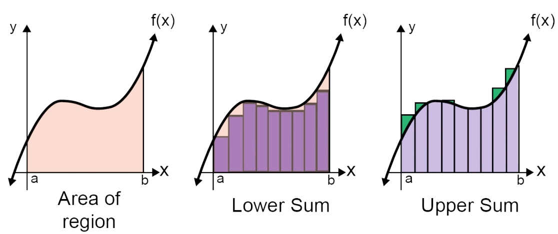
intuitive interpretation: total area of several tiny rectangles.



Calcworkshop.com

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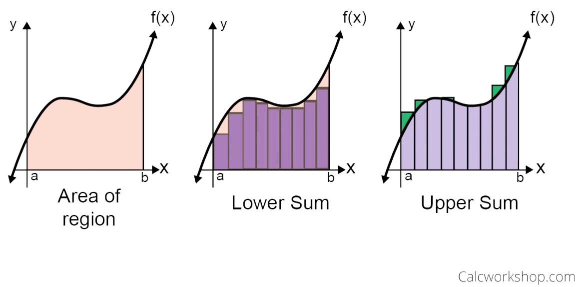


lower sum : pick x_i^* so that $f(x_i^*)$ is smallest.

upper sum : pick x_i^* so that $f(x_i^*)$ largest.

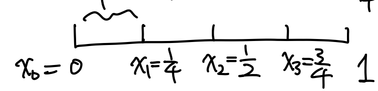
left sum : $x_i^* = x_{i-1}$

right sum : $x_i^* = x_i$.

Problem 6 from Spring 2020 final exam: Consider the integral

$\int_0^1 \sqrt{1+x^2}\,dx.$ $\qquad f(x)=\sqrt{1+x^2}.$

$\frac{1}{4}=\Delta x_1=\Delta x_2=\Delta x_3=\Delta x_4.$



a. Express the integral as a left Riemann sum with 4 sub-intervals of equal width. You may leave your answer as an unspecified sum

b. Is the above Riemann sum in a. greater than or less than the value of the integral? Briefly explain.

a. $\sum_{i=1}^{4} f(x_{i-1})\Delta x_i = f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + f(x_2)\Delta x_3 + f(x_3)\Delta x_4 = \sqrt{1+0^2}\cdot\frac{1}{4}+\sqrt{1+(\frac{1}{4})^2}\cdot\frac{1}{4}+\sqrt{1+(\frac{1}{2})^2}\cdot\frac{1}{4}+\sqrt{1+(\frac{3}{4})^2}\cdot\frac{1}{4}.$

less than.

b. f is increasing: for any $x_{i-1}\le x\le x_i$, $f(x)\ge f(x_{i-1})$. left Riemann sum = lower sum $\le$ actual value of the integral.

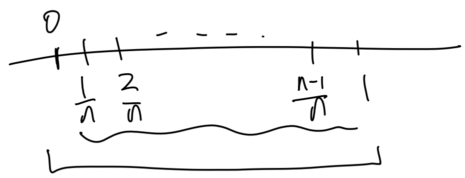
Compute the limit $\lim_{n\to+\infty}\frac{1}{n}\left(\frac{1}{n^2}+\frac{4}{n^2}+\cdots+\frac{(n-1)^2}{n^2}+1\right)$

$\underbrace{\qquad\qquad\qquad}$

sum of n numbers between 0 and 1.

$$= \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{i^2}{n^2} = \lim_{n\to+\infty} \underbrace{\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}}_{f(x)=x^2} = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}\cdot 1 = \frac{1}{3}.$$

(Problem 1 of Spring. Fall 2020 Final) $\ln(n)$. $\lim_{n\to+\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x_i.$



$\frac{n}{n}=1$

$0 \quad \frac{1}{n} \quad \frac{2}{n} \quad \cdots \quad \frac{n-1}{n} \quad 1=\frac{n}{n}.$

$x_0=0,\ x_1=\frac{1}{n},\ \ldots,\ x_{n-1}=\frac{n-1}{n},\ x_n=1.$

$\Delta x_i=\frac{1}{n},\ x_i^*=x_i=\frac{i}{n}.$

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- fundamental theorem of Calculus : $\int_a^b f'(x)\,dx = f(b) - f(a)$;

- substitution law : $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$ $\qquad F(x) = \int f(x)\,dx$.

$$\left(F(g(x))\right)' = F'(g(x)) \cdot g'(x).$$

assume $f(x) = F'(x)$

then $\int f(g(x))\,g'(x)\,dx = \int F'(g(x))\,g'(x)\,dx$

$$= \int \left(F(g(x))\right)'\,dx = F(g(x)).$$

Problem 2.b from Spring 2020 Final: evaluate the integral

$$\int_{-\frac{1}{k}}^{\frac{3}{k}} x\sqrt{kx+1}\,dx,$$ where k is a nonzero constant.

$(x^a)' = a x^{a-1}$

$(\sqrt{x})' = (x^{\frac{1}{2}})'$

$= \frac{1}{2} x^{\frac{1}{2}-1}$

$= \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$$\int_{-\frac{1}{K}}^{\frac{3}{K}} x\sqrt{Kx+1}\;dx$$

$u = f(x)$, then

$du = f'(x)\,dx.$

$f(x) = \sqrt{Kx+1}$

$= \sqrt{h(x)}$ $\quad h(x) = Kx+1$

$f'(x) = \frac{1}{2\sqrt{h(x)}} \cdot h'(x) = \frac{K}{2\sqrt{Kx+1}}$

$-\frac{1}{K} \le x \le \frac{3}{K}$

so $0 \le Kx+1 \le 4$

$0 \le \sqrt{Kx+1} \le 2$

$u = \sqrt{Kx+1}$
$\begin{cases} u^2 = Kx+1, \text{ so } x = \dfrac{u^2-1}{K} \\ du = \dfrac{K}{2\sqrt{Kx+1}}dx, \text{ so } dx = \dfrac{2\sqrt{Kx+1}}{K}du \end{cases}$

via substitution rule,

$\frac{2\sqrt{kx+1}}{k}$ $\sqrt{kx+1} = u.$

$$\int_{-\frac{1}{k}}^{\frac{3}{k}} x\sqrt{kx+1}\,dx = \int_0^2 \frac{u^2-1}{k}\cdot u \left(\frac{2u}{k}\right) du = \frac{1}{k^2}\int_0^2 2u^2(u^2-1)\,du$$

$$= \frac{2}{k^2}\int_0^2 (u^4 - u^2)\,du = \frac{2}{k^2}\left(\frac{1}{5}u^5\Big|_0^2 - \frac{1}{3}u^3\Big|_0^2\right) = \frac{2}{k^2}\left(\frac{1}{5}2^5 - \frac{1}{5}0^0 - \frac{1}{3}2^3 + \frac{1}{3}0^0\right)$$

$$= \frac{2}{k^2}\left(\frac{1}{5}\cdot\overset{2^5}{32} - \frac{1}{3}\cdot\overset{2^3}{8}\right) = \frac{2}{k^2}\left(\frac{96-40}{15}\right) = \frac{2\times 56}{15k^2}.$$

$$\frac{32}{5} - \frac{8}{3} = \frac{32\times 3}{15} - \frac{5\times 8}{15} \qquad = \frac{112}{15k^2}.$$

$$= \frac{96-40}{15}$$

Reminder:

- Quiz 7 on Thursday: Riemann sum (8 pts), Problem 6 of Spring 2020 Final

  Substition law (8 pts), Problem 2 b. of Spring 2020 Fall;

  Evaluating limit (8 pts), Problem 1 b. of Spring 2020 Fall.

- OH: 1-2 pm TODAY. Appointments accepted.

  Grade questions: $\uparrow$ Michael.

See You on Thursday!