Nov. 30th: Review of Chapter 5.

Conceptual Review:

Inverse Function: g is an inverse of f if for any x in the domain of f, $g(f(x)) = x$ (and for any x in dom(g), $f(g(x)) = x$).

Important Results: 1. f continuous on $[a,b] \Longleftrightarrow$ the inverse to f is continuous.

2. f differentiable at a s.t. $f'(a) \neq 0 \Rightarrow (f^{-1})'(f(a)) = \frac{1}{f'(a)}$.

Natural Logarithmic Function: $\ln x = \int_1^x \frac{dt}{t}$, $x > 0$. ($\ln$ 0 ONE) $\log_a x = \frac{\ln x}{\ln a}$. for $a>0$ and $a \neq 1$

Inverse function e^x: **Exponential function.** $a^x = e^{a \ln x}$.

Properties: 1. $\ln(xy) = \ln x + \ln y$, $\ln(x^a) = a \ln x$, $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$;

2. $\lim_{x \to +\infty} \ln x = +\infty$, $\lim_{x \to 0^+} \ln x = -\infty$;

3. $(\ln x)' = \frac{1}{x}$ (by definition)

4. $e^{x+y} = e^x e^y$, $e^{ax} = (e^x)^a$, $e^{x-y} = \frac{e^x}{e^y}$.

5. $(e^x)' = e^x$. 6. $\lim_{x \to +\infty} e^x = +\infty$, $\lim_{x \to -\infty} e^x = 0$.

Review Problems:

Problem 5 [8 points]: Let $F = F(x)$ be the function

$$G(x^3). = F(x) = \underbrace{\int_1^{x^3} \sqrt{t^2+1}\, dt}, \ -\infty < x < +\infty.$$

where $G(x) = \int_1^x \sqrt{t^2+1}\, dt$. $\quad F(1) = 0$.

(a) Explain why the function F is invertible.

(b) Compute $(F^{-1})'(0)$.

Solution. (a) Write $F(x) = G(x^3)$, then $F'(x) = 3x^2 \cdot G'(x^3) = 3x^2 \cdot \sqrt{(x^3)^2+1} = 3x^2\sqrt{x^6+1} \geq 0$ and it has only 1 root $x=0$.

① F is strictly increasing in $(-\infty, 0) \cup (0, +\infty)$, and so in $(-\infty, +\infty)$.

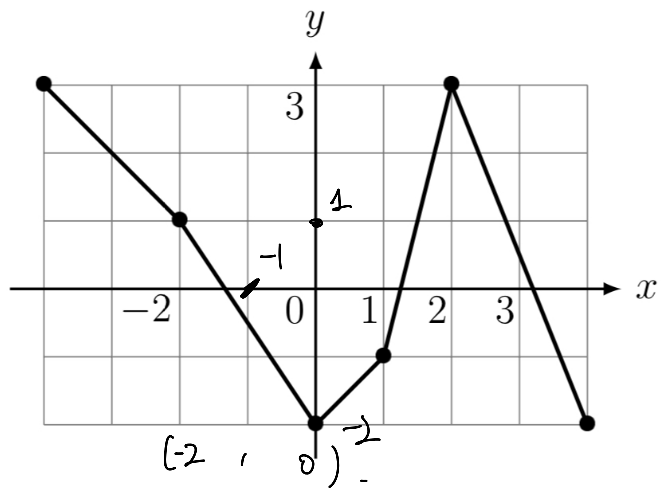
so F is one-to-one and it is invertible.

$f(x) = \int_a^x g(t)\,dt$, then $f(a) = 0$.

(b) we need to find x so that $F(x)=0. \rightsquigarrow x=1$, and $F'(1) = 3\cdot 1^2\sqrt{1^6+1} = 3\sqrt{2} > 0$

$(F^{-1})'(0) = \frac{1}{F'(1)} = \frac{1}{3\sqrt{2}}$. $\square$

Problem 7 [24 points]: Consider the following graph:



if $-2<x<0$, then

$\frac{f(x)+2}{x} = \frac{-2-1}{0+2}$.

$= -\frac{3}{2}$.

$f(x) = -\frac{3}{2}x - 2$.

$f'(x) = -\frac{3}{2}$. for any $-2<x<0$.

(a) Suppose that this is the graph of a function $f = f(x)$. In particular, $f(-1) = -1/2$.

(i) Consider the function $g(x) = f(3x-4)$.
Compute $g'(2)$, if possible. If this is not possible, then explain why.

(ii) Consider the function $h(x) = f\big(f(x)\big) + \ln\big(1-\tan(\pi f(x)/2)\big)$.
Compute $h'(-1)$, if possible. If this is not possible, then explain why.

Solution. (a) (ii). $f(-1) = -\frac{1}{2}$ in $(-2,0)$. so we get

$h'(x) = f'(f(x))\cdot f'(x) + \frac{-\sec^2\left(\frac{\pi f(x)}{2}\right)\cdot \frac{\pi f'(x)}{2}}{1-\tan\left(\frac{\pi f(x)}{2}\right)}$. When $x=-1$, $f(x) = -\frac{1}{2}$.

$\sec^2\left(-\frac{\pi}{4}\right) = \frac{1}{\cos^2\left(-\frac{\pi}{4}\right)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\frac{1}{2}} = 2$.

$\tan\left(-\frac{\pi}{4}\right) = -1$,

$$h'(-1) = \underbrace{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)}_{f'(f(-1))\cdot f'(-1)} + \frac{-\sec^2\left(\frac{\pi}{2}\cdot\left(-\frac{1}{2}\right)\right)\cdot\frac{\pi}{2}\left(-\frac{3}{2}\right)}{1-\tan\left(\frac{\pi}{2}\cdot\left(-\frac{1}{2}\right)\right)} = \frac{9}{4} + \frac{+\sec^2\left(-\frac{\pi}{4}\right)\cdot\left(+\frac{3\pi}{4}\right)}{1-\tan\left(-\frac{\pi}{4}\right)} = \frac{9}{4} + \frac{2\cdot\frac{3\pi}{4}}{1+1} = \underline{\frac{9}{4}+\frac{3\pi}{4}}.$$

$\square$

$g(x) = x^2$, $g(1) = 1$, $g'(x) = 2x$, $\underline{g'(1) = 2}$.

Problem 9 [7 points]: Does the curve $e^y = x + y$ have any points at which the tangent line to the curve is horizontal? Explain your conclusion. (Implicit Differentiation).

Solution: $e^y - x - y = 0$. take derivative w.r.t. x:

$$\frac{d}{dx}(e^y - x - y) = 0$$

$$e^y\frac{dy}{dx} - 1 - \frac{dy}{dx} = 0.$$

$$(e^y - 1)\frac{dy}{dx} = 1, \quad \frac{dy}{dx} = \frac{1}{e^y - 1}$$

horizontal means $\frac{dy}{dx} = 0$, but $\frac{1}{e^y-1}$ is never 0.

so there're no pts at which the tangent line to the curve is horizontal. $\square$

Reminder.

- FINAL Quiz on Thursday

  Practice Problems: the three problems above (or see Fall 2020 Final) and 62 of Section 5.2.

- OH this week: 1-2 pm T, 1-3 pm Th; next week: 10-12 pm Mon, 1-3 pm Tue

- Next Tuesday: Review section (10-12 or 10-11 & 11-12 on Tuesday, Room TBD)