Nov. 30^{th}: Review of Chapter 5.

Conceptual Review:

Inverse Function: g is an inverse of f if $\begin{cases} \text{domain of } g \text{ is range of } f \text{ \& range of } g = \text{domain of } f \\ \text{\& } g(f(x)) = x, \quad f(g(y)) = y. \end{cases}$

Important Results: 1. f continuous on $[a, b] \Longleftrightarrow f^{-1}$ is also continuous.

2. f differentiable at a s.t. $f'(a) \neq 0 \Rightarrow (f^{-1})'(f(a)) = \frac{1}{f'(a)}$.

one criteria: f is strictly increasing/decreasing, then f is invertible (one-to-one).

Natural Logarithmic Function: $\ln x = \int_1^x \frac{dt}{t}$, $x > 0$. $\log_a x = \frac{\ln x}{\ln a}$. $(a>0, a\neq 1)$

Inverse function e^x: **Exponential function.** $a^x = e^{a \ln x}$.

Properties: 1. $\ln(xy) = \ln x + \ln y$, $\ln(x^a) = a \ln x$, $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$;

2. $\lim_{x\to+\infty} \ln x = +\infty$, $\lim_{x\to 0^+} \ln x = -\infty$;

3. $(\ln x)' = \frac{1}{x}$ (by definition)

4. $e^{x+y} = e^x \cdot e^y$, $e^{ax} = (e^x)^a$, $e^{x-y} = \frac{e^x}{e^y}$.

5. $(e^x)' = e^x$ 6. $\lim_{x\to+\infty} e^x = +\infty$, $\lim_{x\to-\infty} e^x = 0$.

Review Problems:

Problem 5 [8 points]: Let $F = F(x)$ be the function

$$G(x^3) = F(x) = \int_1^{x^3} \sqrt{t^2+1}\, dt, \quad -\infty < x < +\infty.$$

where $G(x) = \boxed{\int_1^x \sqrt{1+t^2}\,dt}$. $G(1)=0$.

(a) Explain why the function F is invertible.

(b) Compute $(F^{-1})'(0)$.

Solution. (a) $F'(x) = (x^3)' \cdot G'(x^3) = 3x^2 \cdot \sqrt{(x^3)^2+1} = 3x^2\sqrt{x^6+1}$. $F'(x) \geq 0$ and $F'(x)=0$ only when $x=0$. So F is strictly increasing in $(-\infty, 0) \cup (0, +\infty)$, so in $(-\infty, +\infty)$. So F is invertible.

(b) $(F^{-1})'(F(a)) = \dfrac{1}{F'(a)}$ if $F'(a) \neq 0$. NTSolve. $F(a)=0$, $a=1$. $F'(1) = 3\sqrt{1^6+1} = 3\sqrt{2}$.

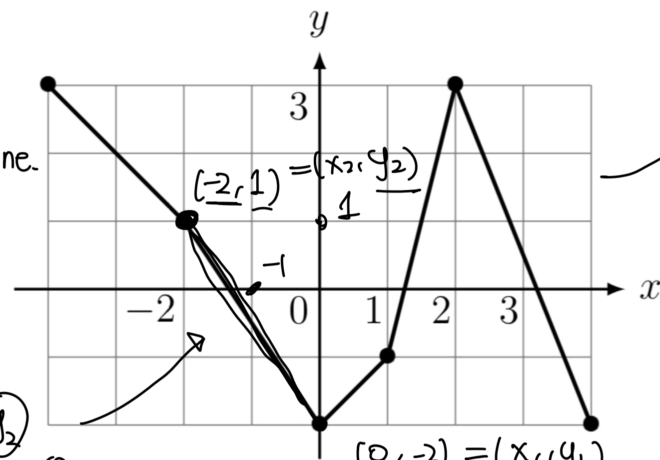
so $(F^{-1})'(0) = \dfrac{1}{F'(1)} = \dfrac{1}{3\sqrt{2}}$. $\square$

Problem 7 [24 points]: Consider the following graph:

derivative of f = slope of this line.

$= \dfrac{y_1 - y_2}{x_1 - x_2}.$

$= \dfrac{-2-1}{0-(-2)}$



$(-2, 1) = (x_2, y_2)$

$(0, -2) = (x_1, y_1)$

$(-2, 0)$

$f(x) - y_2 = k(x - x_2)$.

$f'(x) = \dfrac{-2-1}{0-(-2)} = -\dfrac{3}{2}.$

$f'(-1) = -\dfrac{3}{2}$

$f'(-\tfrac{1}{2}) = -\dfrac{3}{2}$

$f(-1) = -\dfrac{1}{2}$

(a) Suppose that this is the graph of a function $f = f(x)$. In particular, $f(-1) = -1/2$.

(i) Consider the function $g(x) = f(3x - 4)$.
Compute $g'(2)$, if possible. If this is not possible, then explain why.

(ii) Consider the function $h(x) = f\big(f(x)\big) + \ln\big(1 - \tan(\pi f(x)/2)\big)$.
Compute $h'(-1)$, if possible. If this is not possible, then explain why.

Solution. (a) (ii). $h'(x) = f'(f(x)) \cdot f'(x) + \dfrac{-\sec^2\left(\frac{\pi f(x)}{2}\right) \cdot \frac{\pi f'(x)}{2}}{1 - \tan\left(\frac{\pi f(x)}{2}\right)}.$

$h'(-1) = f'(f(-1)) \cdot f'(-1) + \dfrac{-\sec^2\left(\frac{\pi f(-1)}{2}\right) \cdot \frac{\pi f'(-1)}{2}}{1 - \tan\left(\frac{\pi f(-1)}{2}\right)} = f'(-\tfrac{1}{2})(-\tfrac{3}{2}) + \dfrac{-\sec^2\left(\frac{\pi}{2}(-\frac{1}{2})\right) \cdot \frac{\pi}{2}(-\frac{1}{2})}{1 - \tan\left(\frac{\pi}{2}(-\frac{1}{2})\right)}.$

$$= \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right) + \frac{-\sec^2(-\frac{\pi}{4})\cdot(-\frac{\pi}{4})}{1-\underbrace{\tan(-\frac{\pi}{4})}} = \frac{9}{4} + \frac{-\frac{1}{\cos^2\frac{\pi}{4}}(-\frac{\pi}{4})}{1-(-1)} = \frac{9}{4} + \frac{1}{2}\left(-\frac{1}{\frac{1}{2}}\left(-\frac{\pi}{4}\right)\right)$$

$$\cos^2\frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^2 \qquad = \frac{9}{4} + \frac{1}{2}\cdot 2\cdot\frac{\pi}{4} = \frac{9}{4} + \frac{\pi}{4}. \ \square$$

Problem 9 [7 points]: Does the curve $e^y = x + y$ have any points at which the tangent line to the curve is horizontal? Explain your conclusion.

Solution: $e^y - x - y = 0$. take derivative. w.r.t. x.

$$\frac{d}{dx}(e^y - x - y) = 0 \qquad LHS = e^y\frac{dy}{dx} - 1 - \frac{dy}{dx} = 0 = RHS.$$

$$(e^y - 1)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y - 1}.$$ never zero, so no pts s.t. tangent line is horizontal. $\square$

Reminder.

- FINAL Quiz on Thursday — 62 of Section 5.2.

  Practice Problems: the three problems above (or see Fall 2020 Final)

- OH this week: 1-2 pm T, 1-3 pm Th — next week: 10-12 pm Mon, 1-3 pm Tue.

- Next Tuesday: Review section (10-12 or 10-11 & 11-12 on Tuesday, Room TBD)