Final Review Section

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1. Limits

To compute limits, we always start with standard functions and write the limit as combinations of these functions. Explicitly, we assume the following limits:

- $\lim_{x \to x_0} x^a = x_0^a$ for any real number a and any point x_0 so that x_0^a is defined;
- $\lim_{x \to x_0} e^x = e^{x_0}$ for any real number x_0;
- $\lim_{x \to x_0} \ln x = \ln x_0$ for any real number $x_0 > 0$;
- $\lim_{x \to x_0} \sin x = \sin x_0$ and $\lim_{x \to x_0} \cos x = \cos x_0$ for any real number x_0;
- $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

and the following rules of calculating limits:

- $\lim(cf + dg) = c \lim f + d \lim g$, where f, g are functions and c, d are real constants;
- $\lim(fg) = (\lim f)(\lim g)$ where f, g are functions so that their limits exist;
- $\lim\left(\frac{f}{g}\right) = \frac{\lim f}{\lim g}$ where f, g are functions so that their limits exist and the limit of g is non-zero;
- $\lim(f \circ g) = f(\lim g)$ if f, g are functions, f is continuous and the limit of g exists.

There's another way to compute limit: using the definition of derivatives. That is, if a limit is of the form

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

then we can write it as the value of the derivative of f at x, and we can apply rules of derivatives to first compute its derivative and then find the limit.

Here're some practice problems:

1.1 Problem. *Evaluate the given limit:*

i) $\lim_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2}$;

ii) $\lim_{x \to 0} \dfrac{x}{x + \sin(2x)}$;

iii) $\lim_{x\to\infty}(\sqrt{x^2+4x}-x)$;

iv) $\lim_{x\to 1}\dfrac{\ln x-\ln 1}{x-1}$;

v) $\lim_{x\to-\infty}\dfrac{e^{2x}}{e^x+3e^{2x}}$ *and* $\lim_{x\to+\infty}\dfrac{e^{2x}}{e^x+3e^{2x}}$.

1.2 Problem. *Find the derivative of the following function* ***using the definition****. No credit will be given for using derivative rules.*

$$f(x)=\frac{1}{1+x^2}$$

There're important concepts related to limits:

1.3 Definition. A function f is **continuous** at a real number a if $\lim_{x\to a}f(x)=f(a)$.

1.4 Definition. A function f is **differentiable** at a point a if there is a finite real number L so that $\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}=L$.

In exam problems, they'll require us to determine whether a function is continuous and whether a function is differentiable. Sometimes it's just the combination of continuous/differentiable functions so it's continuous/differentiable, and sometimes we need to use definitions of them to conclude.

1.5 Problem. *Consider the following function:(We can change the function to $x^2\sin(\ln x^2)$ if we have an issue with the absolute value, and the parts below still work).*

$$f(x)=\begin{cases} x^2\sin(\ln x^2) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

1. *Show that $f(x)$ is continuous everywhere.*
2. *Find $f'(x)$ for $x\neq 0$.*
3. *Show that $f(x)$ is differentiable at $x=0$.*

1.6 Problem. *Consider the function f defined by*

$$f(x)=\begin{cases} \frac{\sin 5x^2}{x}+8, & \text{if } x<0.\\ (a-b)x+2a, & \text{if } x\geq 0\end{cases}$$

1. *Determine the value of the constant a for which f is continuous at $a=0$. You must carefully justify your answer.*
2. *Determine the values of the constants a and b for which f is differentiable at $x=0$. You must carefully justify your answer.*

2. Derivatives

The computation of derivatives is similar to limits: we start with some standard models

- $(x^a)'=ax^{a-1}$ for any real number a;
- $(\sin x)'=\cos x$ and $(\cos x)'=-\sin x$;
- $(a^x)'=a^x\ln a$ for any $a>0$ but $a\neq 1$;

- $(\log_a x)' = \dfrac{1}{x \ln a}$ for any $a > 0$ but $a \neq 1$.

and using some derivation rules:

- $(cf + dg)' = cf' + dg'$ for any differentiable functions f, g and any constants c, d;
- $(fg)' = f'g + fg'$;
- $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$;
- $\big(f(g(x))\big)' = f'\big(g(x)\big)g'(x)$.

2.1 Problem. *Calculate the derivatives of the following functions:*

1. $f(x) = x^2 \sin(\ln x^2)$;
2. $f(x) = \dfrac{1}{1 + e^{-x}}$;
3. $F(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \frac{\theta}{\theta^4 + 1} \mathrm{d}\theta$;
4. $y = \dfrac{\tan x}{x}$;
5. $f(x) = (1 + x)^{\frac{1}{x}}$.

(2a) Implicit Differentiation. For functions defined implicitly as $F(x, y) = 0$, regarded as a graph of some function $y = f(x)$, we can find the derivative of y using F[1]: take the derivative with respect to x on both sides of the equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x, y) = \frac{\partial F}{\partial x}(x, y) + \frac{\partial F}{\partial y}(x, y) \frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

so we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F / \partial x}{\partial F / \partial y}.$$

Here $\dfrac{\partial F}{\partial x}$ means taking the derivative of F with respect to the variable x, while $\dfrac{\partial F}{\partial y}$ means taking the derivative of F with respect to the variable y.

Implicit differentiation has some geometrical applications: the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the slope of the tangent line to the graph of y, then the implicit differentiation computes the slope of the tangent line to the curve given by the equation $F(x, y) = 0$. With the given slope, we can also determine the equation for the tangent line. When the slope does not exist, by regarding x as a function of y, we can still find the tangent line to this curve.

2.2 Problem. *Does the curve $e^y = x + y$ have any points at which the tangent line to the curve is horizontal? Explain your conclusion.*

2.3 Problem. *Consider the curve given by the equation*

$$\sin(xy) = \cos y + x.$$

Find the tangent line to this curve at the point $(1, \pi)$, and use this to give an estimate of the y-value for a nearby point on the curve where $x = 0.98$.

[1] Here you can ignore all of the following explanations and look at the problems.

(2b) Mean Value Theorems. For a differentiable function f, we have the following mean value theorems:

2.4 Theorem (Fermat)**.** Let f be a function continuous on $[a, b]$ and differentiable in (a, b). If $a < c < b$ is an extreme point of f, then $f'(c) = 0$.

2.5 Theorem (Rolle)**.** Let f be a function continuous on $[a, b]$ and differentiable in (a, b) so that $f(a) = f(b)$, then there is $a < c < b$ such that $f'(c) = 0$.

2.6 Theorem (Mean Value Theorem)**.** Let f be a function continuous on $[a, b]$ and differentiable in (a, b), then there is a real number $a < c < b$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

There's a theorem not quite related to derivatives, but we always combine these results together to solve problems.

2.7 Theorem (Intermediate Value Theorem)**.** If f is continuous in the interval $[a, b]$ and $f(a)f(b) < 0$, then there exists $a < c < b$ such that $f(c) = 0$.

And there's an existence theorem about absolute extrema:

2.8 Theorem. If f is a continuous function on $[a, b]$, then f must have an absolute maximum and an absolute minimum.

Applications of these theorems include finding numbers of roots of a given equation $f(x) = 0$ and proving some inequalities.

2.9 Problem. *Show that* $\sqrt{1+x} \le \sqrt{2} + \dfrac{x-1}{2\sqrt{2}}$ *for* $x \ge 1$.

2.10 Problem. *One wants to calculate the value of the limit*

$$\ell = \lim_{x \to 0+} \frac{\sqrt[3]{5x+64} - 4}{\sqrt[5]{x}}.$$

1. *Show that* $\sqrt[3]{5x+64} < 4 + \dfrac{5}{48}x$ *for all* $x > 0$. *You must justify your methods.*
2. *Use the inequality of part 1. to compute the value of* ℓ. *You may justify your methods.*

2.11 Problem. *Let* $f(x) = x^4 + x - 3$.

1. *Show that* $f(x)$ *has a root in the interval* $[-2, 0]$, *and a root in the interval* $[0, 2]$.
2. *Show that* $f(x)$ *does not have more than two roots.*

2.12 Problem. *Consider the following function defined on* $[-2, 0]$, *by*

$$f(x) = \begin{cases} x^2 + 4x, & -2 \le x \le -1 \\ 2x^2 - 4x - 9, & -1 < x \le 0 \end{cases}$$

1. *Is this function guaranteed to have a global(absolute) maximum and minimum on* $[-2, 0]$*? Carefully justify your answer.*
2. *Find the point* $x = c$ *that satisfies the* ***Mean Value Theorem for Derivatives*** *on* $[-2, 0]$. *If this function has no such point, explain why.*
3. *Find the global* ***minimum*** *and global* ***maximum*** *of* $f(x)$, *if they exist, and also where each occurs.*

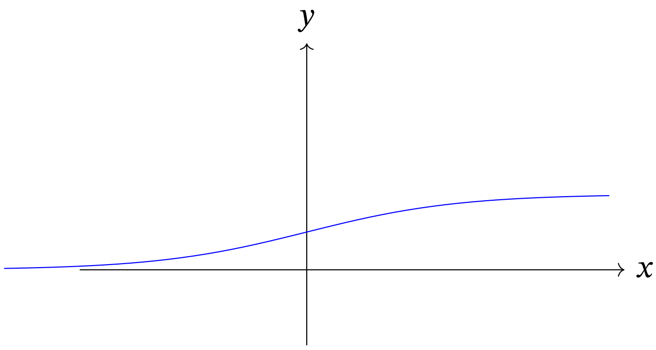
3. Curve Sketching

Given a function $y = f(x)$, we can sketch its graph via the following steps:

1. Find the domain of f;
2. Find the x- and y-intersects of f;
3. Determine horizontal and vertical asymptotes of f;
4. Find the first derivative f' of f, and use the first derivative to determine the interval whether f is increasing or decreasing and critical points of f;
5. Find the second derivative, interval where f is concave up or down, and the inflection points of f;
6. Draw the graph based on the information you get from the above procedure.

We don't need to remember this. Problems in final exams would tell you the procedure by dividing a problem into several subproblems. You only need to follow the line provided by the problem and finally draw the graph.

3.1 Problem. *The graph of $f(x) = \dfrac{1}{1+e^{-x}}$ is given below.*



1. *Prove that f has no vertical asymptotes.*
2. *Prove that f has horizontal asymptotes $y = 0$ and $y = 1$.*
3. *Prove that f is strictly increasing everywhere.*
4. *Prove that f has an inflection point at $x = 0$, and is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.*

3.2 Problem. *Consider the function $f(x) = \dfrac{\sqrt{4-x^2}}{x+1}$ on the domain $[-2, 1) \cup (-1, 2]$. You may freely use any of the following facts.*

i) $f'(x) = \dfrac{-x-4}{(x+1)^2\sqrt{4-x^2}}$;

ii) $f''(x) = \dfrac{(x+1.84)(1.38-x)K(x)}{x+1}$, *where* $K(x) > 0$;

iii) $f(1.38) = 0.6$;

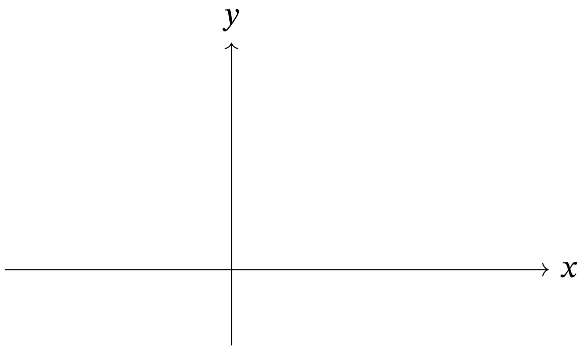
iv) $f(-1.84) = -0.93$.

1. *Study the sign of $f'(x)$. Determine the intervals where f is increasing, and the intervals where it is decreasing. Indicate the values of the local extrema, if any. You must verify your findings.*

2. *Study the sign of $f''(x)$. Determine the intervals where f is concave up, and the intervals where f is concave down. List the inflection points, if any. You must justify your findings.*

3. *Investigate for the existence of vertical/horizontal asymptotes. Your findings must be supported by the careful calculation of relevant limits.(Each vertical asymptote must be supported by two limits.)*

4. *Based on all the information gathered in the previous questions, sketch the graph of f as accurately as possible. Include and clearly label all relavant points and asymptotes.*



3.3 Problem. *Consider the function* $f(x) = \dfrac{(x^4+1)^{\frac{1}{4}}}{1-x}$ *on the domain* $(-\infty, 1) \cup (1, +\infty)$.

1. *Investigate for the existence of horizontal and vertical asymptotes of the graph of f. Your answer must be supported by the careful calculation of relavant lmits.(**Hint:** $(x^4)^{\frac{1}{4}} = |x|$)*

2. $f'(x) = \dfrac{(x+1)(x^2-x+1)}{(1-x)^2(x^4+1)^{\frac{3}{4}}}$. *Note that (x^2-x+1) is always positive. Study the sign of f', then determine the intervals of increase, and of decrease of f. Indicate the values of local extrema, if any.*

3. $f''(x) = \dfrac{(x+1.64)}{1-x}M(x)$, *where $M(x) > 0$. Study the sign of f'', then determine the intervals where f is concave up, and where it is concave down. List all inflection points, if any.*

4. *Based on all the information gathered in the previous questions, sketch the graph of f as accurately as possible. Include all relevant facts as well as some remarkable points.(**Hint:** $2^{\frac{1}{4}} \approx 1.2$; $f(-1.64) \approx 0.65$)*

3.4 Problem. *Let* $f(x) = \dfrac{4x}{x^2+2}$. *The first and second derivatives of $f(x)$ are:*

$$f'(x) = \frac{8-4x^2}{(x^2+2)^2}, \quad f''(x) = \frac{8x(x^2-6)}{(x^2+2)^3}.$$

(You do not need to justify these formulas.)

1. *Give the largest domain on which $f(x)$ is continuous, with a brief explanation on your answer.*

2. *Find all intervals where $f(x)$ is increasing or decreasing, and indicate any local maximums or minimums.*

3. *Find all intervals where $f(x)$ is concave up or concave down, and indicate any inflection points.*

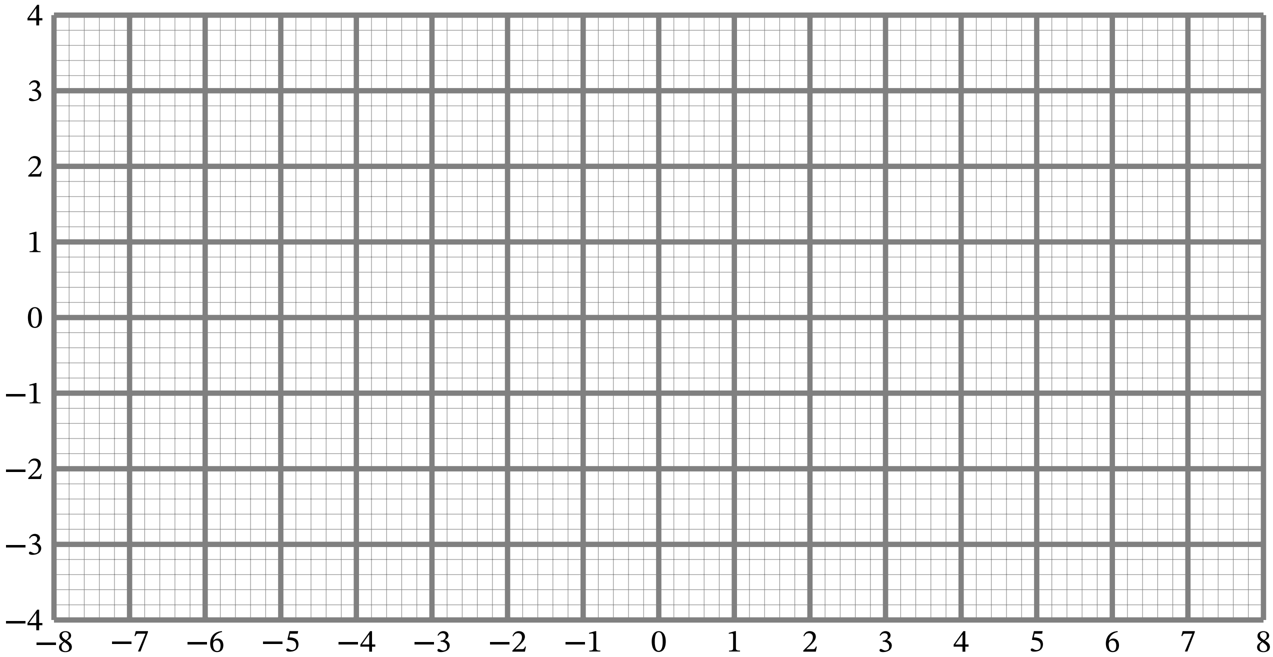


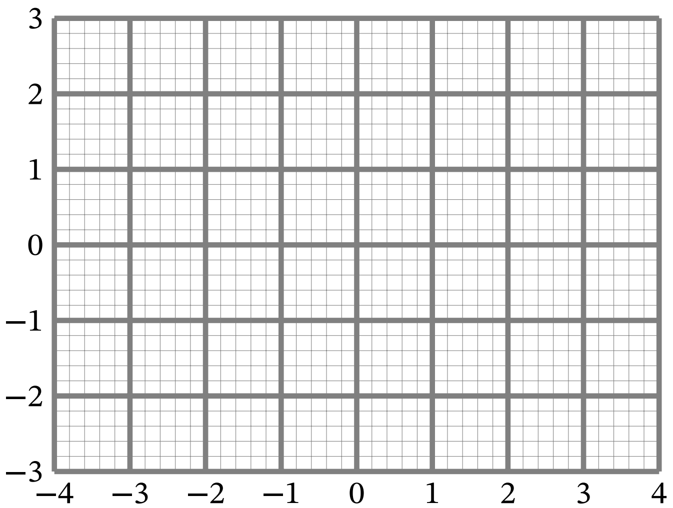
Figure 1: Graph for Problem 3.3

4. *Find any horizontal or vertical asymptotes for the graph $y = f(x)$. Be sure to justify your answer.*

5. *Using your answers to the previous parts, sketch the graph of $f(x)$ on the axes provided, ensuring accuracy of the following features as best you can:*

   - *Intervals where f is increasing/decreasing*
   - *Intervals where $f(x)$ is concave up/down*
   - *Any local maximums or minimums, and inflection points*

   (The coordinates for local extrema and inflection points do not have to be precisely estimated, but should lie between the correct grid lines. For reference, $\sqrt{2} \approx 1.414$ and $\sqrt{6} \approx 2.449$.)



4. Applications

Application problems always give you some background(possibly non-mathematical) and ask you to solve this problem using derivatives or limits. Our strategy is to translate everything into mathematics: we assume some amounts to be the variable and the thing we want to be a function depending on these variables, then we tranlate the problem into the form: given a function $f(x)$, we want to find its derivative at a point/limit/absolute maximum/absolute minimum/etc. Let's look at these problems.

4.1 Problem. *A clothing store produces and sell suits. The total (cumulative) cost, in dollars, of producing q suits is*

$$C(q) = 4000 + 0.25q^2.$$

To sell q suits, the price per suit, in dollars, must be

$$p = 150 - 0.25q.$$

1. *What is the profit from selling q suits?*
2. *How many suits must the store produce and sell to maximize profit?*
3. *What is the maximum profit?*
4. *What price per suit must the store charge to maximize profit?*

4.2 Problem. *The radius r of a right circular cone is* increasing *at the rate of 1 foor per second, and the hight h of the cone is* decreasing *at the rate of 2 feet per second.*

At a particular time, the radius of the cone is 30 feet and the height of the cone is 20 feet. Is the volume of the cone increasing or decreasing at that moment? Explain your conclusion.

4.3 Problem. *It's a hot day in L. A. and Carina has an ice cream cone. The ice cream is leaking into the cone at a rate of $3/2cm^3$ per second. Given that the cone is $10cm$ high, with a radius at the largest end of $3cm$, at the moment when the leaked ice cream fills half-way down the cone, what is the rate of change of the height of the liquid ice cream in the cone?(Hint: the formula for the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the cone, and h is the height.)*

4.4 Problem. *Let x, y, z be the lengths of the sides of a right triangle, where z is the length of the hypothenuse. Assume that x decreases at a rate of $\sqrt{3}$ ft/min, and y increases at a rate of 4 ft/min. At time $t = 0$, we know that $x = 2$ ft and $y = 2\sqrt{3}$ ft.*

1. *Find the rate of change of z at time $t = 0$. Your answer must be fully simplified.*
2. *Let θ be the angle between the sides x and z. Find the value of θ at time $t = 0$.*
3. *Find the rate of change of θ at time $t = 0$. Your answer may be fully simiplified.*

4.5 Problem. *A 6-foot-tall man wqalks at the speed of 7 ft/sec straight towards a streetlight that is 20 feet high.*

1. *At what rate is the length of his shadow changing?*
2. *At what rate is the tip of his shadow moving along the ground?*

4.6 Problem. *A factory wants to build cylindrical office waste bins with a hemispherical cap on the top and a disk at the bottom. Each bin is made of 6π square feet of material. What is the maximal capacity of a bin? You must carefully justify your methods.*

*(**Hint:** Area of a cylinder = (circumference)(height) = $2\pi rh$*
Volume of a cylinder = (area)(height) = $\pi r^2 h$
Area of a sphere = $4\pi r^2$
Volume of a sphere = $\frac{4}{3}\pi r^3$)

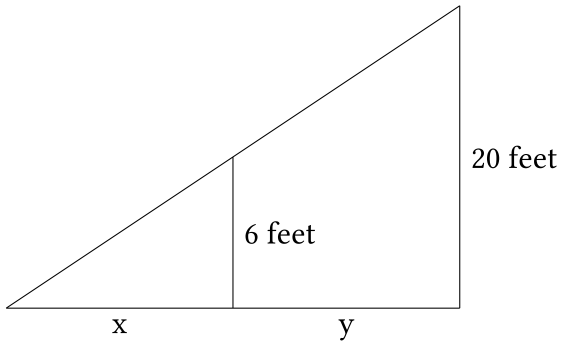


Figure 2: Picture for Problem 4.5

4.7 Problem. *A deposit of ore contains 100-mg of radium-226, which undergoes radioactive decay. After 500 years, 80.4% of the original mass of radium-226 remains.*[2]

1. *Find the mass* $m(t)$ *of radium-226 that remains after* t *years.*
2. *What is the half-life of radium-226?*
3. *When will there be 20-mg of radium-226 remaining?*

[2]Constants in your answers to this problem may be left in exact form, e.g. $1.234 \cdot \ln(5.678)$.