Math 125 - Fall 2021	Tuesday, October 5th, 11am

Midterm 1 – Part I

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Instructions

- You are allowed one half-page (front and back) of handwritten notes.
- All cell phones, calculators, and other devices must be turned off and out of reach.
- Responses should include step-by-step work with relevant justification, taking note of problem instructions. Answers without much explanation may not receive much credit.
- Please write clearly and legibly, and when asked to perform a calculation, put a box around your final answer.

Problem	Score
0	/ 2
1	/ 24
2	/ 24
Total	/ 50

Problem 0. Name someone who has had a significant influence on your life and say how. (2 points)

Problem 1.

Evaluate any TWO of the limits below, circling the ones you choose to complete. If the limit does not exist, say whether it is $+\infty$, $-\infty$, or neither. You may use any method covered thus far,[1] but you should provide full justification for your conclusion. (12 points each)

a) $\displaystyle\lim_{x\to -2} \frac{x+2}{x^2+3x+2}$ b) $\displaystyle\lim_{x\to 0} \frac{\tan(2x)-\sin(3x)}{x}$ c) $\displaystyle\lim_{x\to\infty} \frac{1}{x+\sin(x)}$

[1] This does not include l'Hôpital's rule.

Solution. a) $\displaystyle\lim_{x\to -2} \frac{x+2}{x^2+3x+2} = \lim_{x\to -2} \frac{x+2}{(x+2)(x+1)} = \lim_{x\to -2} \frac{1}{x+1} = \frac{1}{-2+1} = -1;$

b) $\displaystyle\lim_{x\to 0} \frac{\tan(2x)-\sin(3x)}{x} = \lim_{x\to 0} \frac{\sin(2x)-\sin(3x)\cos(2x)}{x\cos(2x)}$

$$= \lim_{x\to 0} \frac{\sin(2x)}{x\cos(2x)} - \lim_{x\to 0} \frac{\sin(3x)\cos(2x)}{x\cos(2x)} = \lim_{x\to 0} \frac{\sin(2x)}{2x} \lim_{x\to 0} \frac{2}{\cos(2x)} - \lim_{x\to 0} \frac{\sin(3x)}{3x}\cdot 3$$

$$= 1\cdot \frac{2}{\cos 0} - 1\cdot 3 = 2-3 = -1.$$

c) Note that $-1 \le \sin x \le 1$, hence we have

$$x-1 \le x+\sin x \le 1+x,$$

and so

$$\frac{1}{x+1} \le \frac{1}{x+\sin x} \le \frac{1}{x-1}$$

if $x>1$. By taking limit as $x\to+\infty$, we have

$$\lim_{x\to+\infty} \frac{1}{x+1} = \lim_{x\to-\infty} \frac{1}{x-1} = 0,$$

therefore by the squeeze theorem, $\displaystyle\lim_{x\to+\infty} \frac{1}{x+\sin x} = 0.$

(SPACE FOR PROBLEM 1)

Problem 2. (a) (12 points) Find a value of c such that the limit exists:

$$\lim_{x\to 2} \frac{\frac{4}{x} - c}{x - 2}$$

Evaluate the limit by direct computation. Explain why the value of c is unique, i.e. why the limit does not exist for any other value of c.

(b) (6 points) Find a function $f(x)$ and a point $x = a$ such that the limit in part (a) equals $f'(a)$, and verify that the value agrees with a standard formula for $f'(x)$.

(c) (6 points) Let c be as in part (a) and define a new function by $q(x) = \dfrac{f(x) - c}{x - 2}$.

Give the largest domain where the above defines a continuous function. Does $q(x)$ extend continuously to a larger domain? Explain why or why not.

Solution. (a) We pick $c=2$, then $\lim_{x\to 2} \frac{\frac{4}{x}-2}{x-2} = \lim_{x\to 2} \frac{4-2x}{x(x-2)} = \lim_{x\to 2} \frac{2(2-x)}{x(x-2)}$

$= \lim_{x\to 2}\left(-\frac{2}{x}\right) = -\frac{2}{2} = -1.$

This value is unique because for the limit to exist, note that $\lim_{x\to 2}(x-2)=0$, we must have $\lim_{x\to 2}\left(\frac{4}{x}-c\right)=0$, hence $2-c=0$ and therefore $c=2$.

(b) (The choice of f is NOT unique) We pick $f(x)=\frac{4}{x}$ and $a=2$, then $f'(x) = -\frac{4}{x^2}$ and $f'(2) = -\frac{4}{2^2} = -\frac{4}{4} = -1 = \lim_{x\to 2} \frac{\frac{4}{x}-2}{x-2}$.

(c) $q(x) = \frac{\frac{4}{x}-2}{x-2} = \frac{4-2x}{x(x-2)}$, so q is defined in the interval $(-\infty,0)\cup(0,2)\cup(2,+\infty)$.

Since q is rational, q is continuous in its domain. Since $\lim_{x\to 2} \frac{4-2x}{x(x-2)} = -1$, there's a continuous extension $\tilde{q}$ of q to the region $(-\infty,0)\cup(0,+\infty)$ such that

$$\tilde{q}(x) = \begin{cases} q(x), & \text{when } x\in(-\infty,0)\cup(0,2)\cup(2,+\infty); \\ -1, & \text{when } x=2. \end{cases}$$

But since $\lim_{x\to 0^-} q(x) = +\infty$, $\lim_{x\to 0^+} q(x) = -\infty$, $\lim_{x\to 0} q(x)$ does not exist, hence $(-\infty,0)\cup(0,+\infty)$ is the largest domain that $q(x)$ can extend.

(SPACE FOR PROBLEM 2)