Contents

Midterm Part I Problem 1

Let $f(x) = \frac{1}{4}x^4 - x^3$.

(a)(12 points) Find all critical points of $f(x)$ and list all intervals on which $f(x)$ is increasing or decreasing. Classify each point as a local maximum, local minimum, or neither.

Solution. The first derivative of $f(x)$ is $f'(x) = x^3 - 3x^2 = x^2(x-3)$, so the equation $f(x) = 0$ has two solutions $x_1 = 0$ and $x_2 = 3$. We divide $(-\infty, \infty)$ into three parts: $(-\infty, 0),\ (0, 3)$ and $(3, +\infty)$.

- For $x < 0$, we have $x^2 > 0$ and $x - 3 < 0$, hence $f'(x) = x^2(x-3) < 0$, so f is decreasing in $(-\infty, 0)$;
- For $0 < x < 3$, $x^2 > 0$ and $x - 3 < 0$, so we have $f'(x) = x^2(x-3) < 0$ and hence f is decreasing in $(0, 3)$. Combined with the first one, we have $f(x)$ is decreasing in the interval $(-\infty, 3)$;
- For $x > 3$, $x^2 > 0$ and $x - 3 > 0$, so we have $f'(x) = x^2(x-3) > 0$ and hence $f(x)$ is increasing in $(3, +\infty)$.

We have two critical points 0 and 3, and by the first derivative test, 3 is a local minimum of f, while 0 is neither a local maximum nor a local minimum.$\diamond$

(b)(6 points) Find all inflection points for $f(x)$, and list all intervals on which $f(x)$ is concave up or concave down.

Solution. The second derivative of $f(x)$ is $f''(x) = 3x^2 - 6x = 3x(x-2)$, and the equation $f''(x) = 0$ has two solutions: $x_1 = 0$ and $x_2 = 2$. According to the data above, we can divide $(-\infty, +\infty)$ into three parts:

- For $x < 0$, $x - 2 < 0$, so we have $f''(x) > 0$ and $f(x)$ is concave up in $(-\infty, 0)$;
- For $0 < x < 2$, $x - 2 < 0$, so we have $f''(x) < 0$ and $f(x)$ is concave down in $(0, 2)$;
- For $x > 2$, $x - 2 > 0$ and $x > 0$ so $f''(x) > 0$ and $f(x)$ is concave up in $(2, +\infty)$,

and the inflection points of $f(x)$ are just 0 and 3.$\diamond$

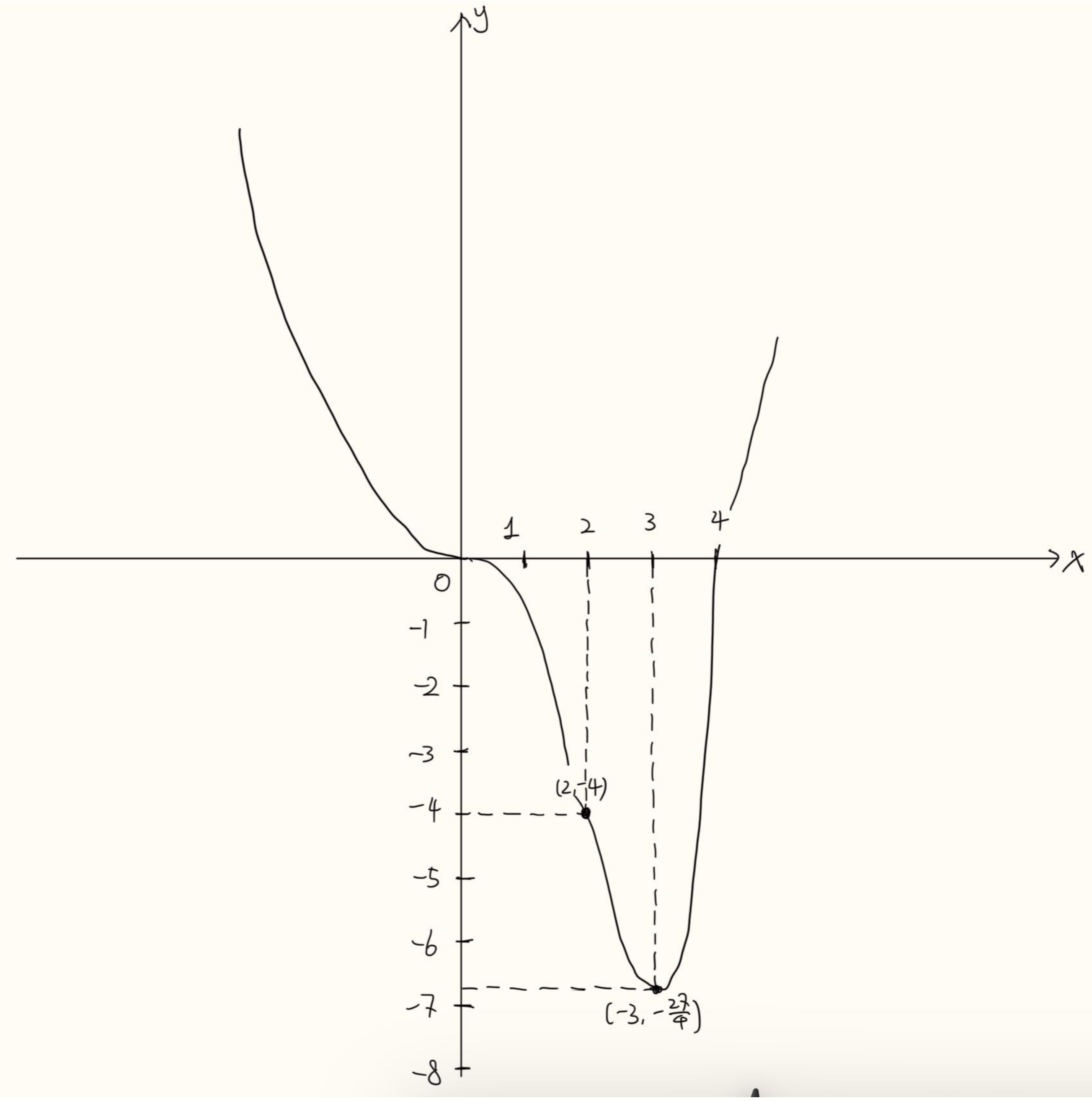
(c)(6 points) Sketch the graph of $f(x)$ on the axis provided.

Your sketch should reflect the following features(as best as you can draw them):

- Critical points

- Intervals where $f(x)$ is increasing/decreasing
- Inflection points
- Intervals where $f(x)$ is concave upwards/downwards

Solution.



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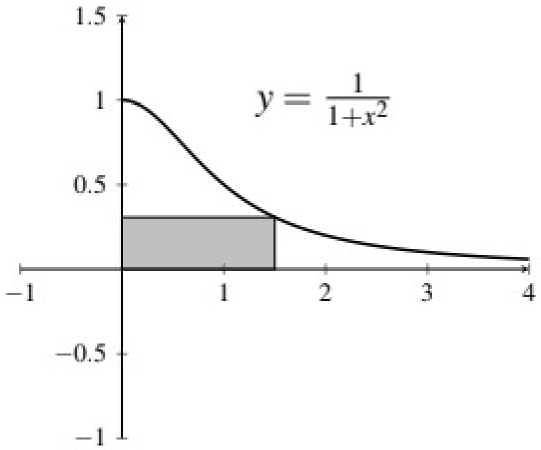
Midterm Part I Problem 2

(12 points) A rectangle in the first quadrant is inscribed between the x-axis, the y-axis, and $y = \dfrac{1}{1+x^2}$ as pictudarkblue. Find the maximum value for its area, with justification. If the maximum value does not exist, explain why.



Solution. The area of the rectangle is given by $A(x) = xy = \dfrac{x}{1+x^2}$, and its derivative is

$$A'(x) = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}. \tag{12}$$

The equation $A'(x) = 0$ has two solutions $x_1 = 1$ and $x_2 = -1$, but the domain of A is $[0, \infty)$,(It cannot be negative since then the function makes no sense) so A has only one critical point $x = 1$. We divide the domain into two parts: $[0, 1)$ and $(1, +\infty)$:

- For $0 \le x < 1$, we have $1 + x^2 > 0$ and $1 - x^2 > 0$, so $A'(x) > 0$ and $A(x)$ is increasing in $[0, 1)$;
- For $1 < x < +\infty$, $1 + x^2 > 0$ while $1 - x^2 < 0$, so $A'(x) < 0$ and $A(x)$ is decreasing in $(1, +\infty)$.

Therefore by the first derivative test, $x = 1$ is a local maximum of $A(x)$, and since it's the only critical point of $A(x)$ in $[0, +\infty)$, $x = 1$ is the absolute maximum of $A(x)$, so the maximum value for its area is $A(1) = \frac{1}{2}$.$\diamond$

Midterm Part I Problem 3

(12 points) Find the number of solutions to the equation $x^4 = \sqrt{1-x^2}$.

Solution. Note that solutions to the equation $x^4 = \sqrt{1-x^2}$ inside the interval $[-1, 1]$ are exactly solutions to the equation $x^8 = 1 - x^2$, so we only need to find solutions for the equation $x^8 = 1 - x^2$. Write $f(x) = x^8 + x^2 - 1$, then we have $f'(x) = 8x^7 + 2x = 2x(4x^6 + 1)$. Note that $4x^6 + 1 > 0$ for any x, so we have

- For $x < 0$, $f'(x) < 0$, hence $f(x)$ is decreasing in $[-1, 0)$;
- For $x > 0$, $f'(x) > 0$, hence $f(x)$ is increasing in $(0, 1]$.

Therefore $f(x)$ has at most one root in each of these two intervals. Now $f(0) = -1 < 0$, and we have $f(-1) = f(1) = 1 + 1 - 1 = 1$, hence by the intermediate value theorem, $f(x)$ has one root in $(-1, 0)$ and another root in $(0, 1)$, therefore $f(x)$ has exactly two roots in $[-1, 1]$, and therefore the equation $x^4 = \sqrt{1-x^2}$ has exactly two solutions in $[-1, 1]$.$\diamond$

Midterm Part II Problem 1

Consider the integral

$$\int_0^3 \sqrt{9 + x^3}\mathrm{d}x. \tag{13}$$

(a)(12 points) Calculate Riemann sum approximation as $n = 3$ equal sub-intervals using both left and right endpoints.(You may leave your answers unsimplified.)

Explain how to decide if each is an underestimate or overestimate.

Solution: When $n = 3$, the length of each sub-interval is $\frac{3}{n} = \frac{3}{3} = 1$, and these three intervals are $[0, 1]$, $[1, 2]$ and $[2, 3]$. Hence the left Riemann sum is

$$L_3 = 1 \cdot \sqrt{9 + 0^3} + 1 \cdot \sqrt{9 + 1^3} + 1 \cdot \sqrt{9 + 2^3}, \tag{14}$$

and the right Riemann sum is

$$R_3 = 1 \cdot \sqrt{9 + 1^3} + 1 \cdot \sqrt{9 + 2^3} + 1 \cdot \sqrt{9 + 3^3}. \tag{15}$$

Write $f(x) = \sqrt{9 + x^3}$. Since $f'(x) = \frac{3x^2}{2\sqrt{9+x^3}} > 0$ when $x > 0$, the function f is strictly increasing in the interval $(0, 3)$, hence L_3 is always a lower sum and R_3 is an upper sum, therefore L_3 is an underestimate and R_3 is an overestimate.

(Or if you want to explain more explicitly, you can say $\int_0^3 f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^2 f(x)\mathrm{d}x + \int_2^3 f(x)\mathrm{d}x \geq 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = L_3$ and similarly for R_3.)$\diamond$

(6 points) What value of n is sufficient to guarantee that a Riemann sum approximation with n equal sub-intervals is accurate to within 10^{-3}?

Solution: It's something we didn't cover in discussion section and requidarkblue some understanding of the Riemann sum formula. In general, the largest possible error of the Riemann sum approximation is the difference of the upper sum and the lower sum, and in this case, the upper sum is the right sum R_n and the lower sum is the left sum L_n, so we must have

$$|R_n - L_n| \leq 10^{-3}, \tag{16}$$

since $R_n \geq L_n$, this is equivalent to $R_n - L_n \leq 10^{-3}$, and we can compute this difference out as:

$$R_n - L_n = \frac{3}{n}\sum_{k=1}^{n} f\left(\frac{3(k-1)}{n}\right) - \frac{3}{n}\sum_{k=1}^{n} f\left(\frac{3k}{n}\right) = \frac{3}{n}(f(3) - f(0)) = \frac{9}{n}. \tag{17}$$

Therefore condition (4) is the same as $\frac{9}{n} \leq 10^{-3}$, and hence $n \geq 9 \times 10^3 = 9000$.$\diamond$

(b)(6 points) Explain why the integral equals

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{9 + \left(\frac{3k}{n}\right)^3} \left(\frac{3}{n}\right). \tag{18}$$

Solution: From theorem 4 of the textbook, we know that

$$\int_0^3 \sqrt{9 + x^3}\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \tag{19}$$

where $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $x_k = 0 + k\Delta x = \frac{3k}{n}$ and $f(x) = \sqrt{9 + x^3}$. Therefore RHS in (7) is equal to

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{9 + \left(\frac{3k}{n}\right)^3} \left(\frac{3}{n}\right), \tag{20}$$

which is exactly the limit given in the problem.

For this problem, you need at least to relate all the terms in this given limit to the general formula presented in the textbook, but you don't need to mention the number of the theorem.$\diamond$

Midterm Part II Problem 2

Calculate each integral.

(a) (12 points) $\int_0^1 \frac{4x+2}{(x^2+x+3)^3}\mathrm{d}x$

Solution. Let $u = x^2 + x + 3$, then $\mathrm{d}u = (2x+1)\mathrm{d}x$ and since $0 \le x \le 1$, we have $3 \le u \le 5$, so by substitution law, we have

$$\int_0^1 \frac{4x+2}{(x^2+x+3)^3}\mathrm{d}x = \int_3^5 \frac{2(2x+1)}{u^3}\frac{\mathrm{d}u}{(2x+1)} = \int_3^5 \frac{2\mathrm{d}u}{u^3} = -\frac{1}{u^2}\bigg|_3^5 = \frac{1}{9} - \frac{1}{25} = \frac{16}{225}. \diamond \tag{21}$$

(b) (12 points) $\int x\sin\left(x^2\right)\sin\left(2+\cos\left(x^2\right)\right)\mathrm{d}x$

Solution. Let $u = 2 + \cos\left(x^2\right)$, we have $\mathrm{d}u = -2x\sin\left(x^2\right)\mathrm{d}x$, and using substitution law, we have

$$\begin{aligned}\int x\sin\left(x^2\right)\sin\left(2+\cos\left(x^2\right)\right)\mathrm{d}x &= \int x\sin\left(x^2\right)\sin u\frac{\mathrm{d}u}{-2x\sin x^2} = -\frac{1}{2}\int \sin u\mathrm{d}u = \frac{1}{2}\cos u + C \\ &= \frac{1}{2}\cos\left(2+\cos\left(x^2\right)\right) + C. \diamond\end{aligned} \tag{22}$$