Name: ______________________________ **MATH 125**
Date: 11/04 **Quiz 6A**

Problem 1. (8 points) Find two real numbers such that the sum of three times the first number and the second is 30 and the product of the numbers is as large as possible.

Solution: Assume these two numbers are x and y, then we have the equation

$$3x+y=30 \qquad (1)$$

Let $p=xy$ be their product. We want p to be as large as possible. (1) reads

$$y=30-3x.$$

Plugging in p, we get $p(x)=x(30-3x)=-3x^2+30x$. Hence $p'(x)=-6x+30$. The equation $p'(x)=0$ has only one solution $x=5$, and since $p''(x)=-6<0$, $x=5$ is a local maximum of p. Since it's the only critical pt of p, $x=5$ is the global maximum of p, and in this case, $y=30-3x=30-3\cdot 5=30-15=15$, hence 5 is the first number and 15 is the second. □

Problem 2. (8 Points) Show that the equation $x+\sin x=0$ has exactly one root.

Proof. Let $f(x)=x+\sin x$, then we have $f'(x)=1+\cos x\geqslant 0$. Since $f(-\pi)=-\pi<0$ and $f(\pi)=\pi>0$, by intermediate value theorem, f has at least one root in $(-\pi,\pi)$. Now assume f has at least two roots, then because $f'(x)\geqslant 0$ for any x, f is increasing, so for any $x\leq -\pi$, $f(x)\leq f(-\pi)=-\pi<0$ and for any $x\geqslant\pi$, $f(x)\geqslant f(\pi)=\pi>0$, so another root of f must lie in the interval $(-\pi,\pi)$. By Rolle's theorem, $f'(x)$ must have a root in $(-\pi,\pi)$, but $f(x)=1+\cos x$ has no roots in $(-\pi,\pi)$, a contradiction.

Therefore, $f(x)$ has at most one root and hence exactly one root. □

Problem 3. (8 points) Find f for the following derivatives and initial values:
(i) $f'(x)=\sqrt{x+1}$ with $f(1)=1$; (ii) $f'(x)=\sin(2x)$ with $f(0)=\frac{1}{2}$.

Solution. (i) Since $f'(x)=\sqrt{x+1}=(x+1)^{\frac{1}{2}}$, we have $f(x)=\frac{2}{3}(x+1)^{\frac{3}{2}}+C$. $f(1)=1$ implies that $\frac{2}{3}(1+1)^{\frac{3}{2}}+C=1$, so $C=1-\frac{2}{3}\times 2^{\frac{3}{2}}=1-\frac{2}{3}\cdot\sqrt{2^3}=1-\frac{4}{3}\sqrt{2}$. Therefore $f(x)=\frac{2}{3}(x+1)^{\frac{3}{2}}+1-\frac{4}{3}\sqrt{2}$.

(ii) Since $(\cos 2x)'=-2\sin 2x$, we have $f(x)=-\frac{1}{2}\cos 2x+C$. $f(0)=\frac{1}{2}$ implies $-\frac{1}{2}+C=\frac{1}{2}$, and hence $C=1$. Therefore $f(x)=-\frac{1}{2}\cos 2x+1$. □

Final Score: ______________

Name: ______________________________ **MATH 125**

Date: 11/04 **Quiz 6B**

Problem 1. (8 points) Find two real numbers such that the sum of two times the first number and five times the second is 10 and the product of the numbers is as large as possible.

Solution. Assume these two #s are x and y, then they should satisfy

$$2x+5y=10 \qquad (1)$$

which implies $y=2-\frac{2}{5}x$. Let $p=xy$ be the product, plugging in y gives $p(x)=xy=x(2-\frac{2}{5}x)=-\frac{2}{5}x^2+2x$.

Hence $p'(x)=-\frac{4}{5}x+2$ and $p''(x)=-\frac{4}{5}<0$. $p'(x)=0$ has only one solution $x=\frac{5}{2}$, and since $p''(x)<0$, $x=\frac{5}{2}$ is a local maximum and an absolute maximum since it's the only critical point. $y=2-\frac{2}{5}x=2-\frac{2}{5}\cdot\frac{5}{2}=2-1=1$, therefore the first number is $\frac{5}{2}$ and the second is 1. □

Problem 2. (8 Points) Show that the equation $x-\cos x=0$ has exactly one root.

Proof. Let $f(x)=x-\cos x$, then $f'(x)=1+\sin x\geq 0$, so f is (not necessarily strictly) increasing everywhere. Since $f(-\frac{\pi}{2})=-\frac{\pi}{2}<0$ and $f(\frac{\pi}{2})=\frac{\pi}{2}>0$, by IVT, f has at least one root in $(-\frac{\pi}{2},\frac{\pi}{2})$. For $x\leq-\frac{\pi}{2}$, $f(x)\leq f(-\frac{\pi}{2})<0$ and for $x\geq\frac{\pi}{2}$, $f(x)\geq f(\frac{\pi}{2})>0$, so all roots of f lie in $(-\frac{\pi}{2},\frac{\pi}{2})$, and for any $-\frac{\pi}{2}<x<\frac{\pi}{2}$, $1+\sin x>1+\sin(-\frac{\pi}{2})=1-1=0$, hence f strictly increasing in $(-\frac{\pi}{2},\frac{\pi}{2})$, so f has at most 1 root in $(-\frac{\pi}{2},\frac{\pi}{2})$, and therefore f has exactly one root. □

Problem 3. (8 points) Find f for the following derivatives and initial values:

(i) $f'(x)=\sqrt{x+4}$ with $f(-3)=1$; (ii) $f'(x)=\cos(2x)$ with $f(0)=1$.

Solution. (i) $f(x)=\frac{2}{3}(x+4)^{\frac{3}{2}}+C$, since $f(-3)=1$, we have $\frac{2}{3}(-3+4)^{\frac{3}{2}}+C=\frac{2}{3}+C=1$, hence $C=\frac{1}{3}$.

Therefore $f(x)=\frac{2}{3}(x+4)^{\frac{3}{2}}+\frac{1}{3}$.

(ii) $f(x)=\frac{1}{2}\sin(2x)+C$ with $f(0)=\frac{1}{2}\sin 0+C=C=1$, hence $C=1$ and $f(x)=\frac{2}{3}(x+4)^{\frac{3}{2}}+1$. □

Final Score: Siyang Lin